Module MAU23203: Analysis in Several Real Variables
Michaelmas Term 2021

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Contents

A Dedekind Sections and the Real Number System

Definition For the purposes of this discussion of the construction of the real number system using Dedekind sections, we say that a subset α of the set $\mathbb{Q}$ of rational numbers is a *left segment* of the rational numbers if and only if it satisfies the following four conditions:

(i) $\alpha \neq \emptyset$ (i.e., α is non-empty);

(ii) $\alpha \neq \mathbb{Q}$ (i.e., α is not the entire set of rational numbers);

(iii) if p and q are rational numbers, if $q \in \alpha$, and if $p < q$, then $p \in \alpha$;

(iv) if q is a rational number, and if $q \in \alpha$, then there exists a rational number r such that $r > q$ and $r \in \alpha$.

Each rational number q determines a corresponding left segment λ_q of the rational numbers, defined so that

$$\lambda_q = \{p \in \mathbb{Q} : p < q\}.$$

We refer to the left segment λ_q as the left segment of the rational numbers that *represents* the rational number q.

Example Let α be the subset of the rational numbers defined such that

$$\alpha = \{p \in \mathbb{Q} : p \leq 0 \text{ or } p^2 < 2\}.$$

Now α is a left segment of the rational numbers. We prove that there is no rational number q for which $\alpha = \lambda_q$. (In other words, we prove that the left segment α does not represent any rational number.) For this purpose we make use of the fact that there is no rational number q satisfying $q^2 = 2$.

Let q be a rational number that does not belong to the left segment α. Then $q > 0$ and $q^2 > 2$. Let p be a rational number. Then

$$p^2 - q^2 + 2q(q - p) = p^2 + q^2 - 2pq = (q - p)^2 \leq 0$$

and therefore

$$p^2 \geq q^2 - 2q(q - p).$$

It follows that if p is a rational number satisfying the inequalities

$$q - \frac{q^2 - 2}{2q} < p < q$$

then $2q(q-p) < q^2 - 2$ and therefore $p^2 > 2$. But then $p \in \lambda_q$ and $p \notin \alpha$. We conclude that the left segment α does not represent any rational number q that does not belong to α. Moreover no left segment represents any of the rational numbers that belong to it. We conclude therefore that the left segment α of the rational numbers does not represent any rational number.

Lemma A.1 *Let α be a left segment of the rational numbers and let q and r be rational numbers. Suppose that $q \in \alpha$ and $r \notin \alpha$. Then $q < r$.*

Proof It cannot be the case that $q = r$. If it were the case that $r < q$ then, given that $q \in \alpha$, it would follow from property (iii) in the definition of left segments that $r \in \alpha$, which is not the case. Therefore the rational number r cannot be either equal to or less than the rational number q. Therefore $q < r$, as required. ∎

Lemma A.2 *Let α be a left segment of the rational numbers. Then, given any positive rational number e, there exist rational numbers v and w such that $v \in \alpha$, $w \notin \alpha$ and $0 < w - v < e$.*

Proof Let α be a left segment of the rational numbers, and let a positive rational number e be given. It follows from properties (i) and (ii) in the definition of left segments that there exist rational numbers s and t such that $s \in \alpha$ and $t \notin \alpha$. Moreover it follows from property (iii) in that definition that $t > s$. Choose a positive integer M large enough to ensure that $Me > t - s$, let k be the largest non-negative integer for which

$$s + \frac{k(t-s)}{M} \in \alpha$$

and let

$$v = s + \frac{k(t-s)}{M} \quad \text{and} \quad w = s + \frac{(k+1)(t-s)}{M}.$$

Then $s \in \alpha$, $w \notin \alpha$ and $0 < w - v < e$, as required. ∎

A.1 The Ordering of Left Segments

Lemma A.3 *Let α and β be left segments of the rational numbers, and let r be a rational number. Suppose that $r \in \beta$ and $r \notin \alpha$. Then $\alpha \subset \beta$.*

Proof Let q be a rational number belonging to α. If it were the case that $q > r$ then property (iii) in the definition of left segments would require the rational number r to belong to the set α, which is not the case. Therefore q is not greater than r. Also q is not equal to r, because $q \in \alpha$ and $r \notin \alpha$.

Therefore $q < r$. Now $r \in \beta$. It therefore follows from property (iii) in the definition of left segments that $q \in \beta$. We conclude therefore that every rational number belonging to the left segment α also belongs to the left segment β. Thus $\alpha \subset \beta$, as required. ∎

Definition Let α and β be left segments of the rational numbers. We write $\alpha < \beta$ if and only if both $\alpha \subset \beta$ and $\alpha \neq \beta$, in which case we say that the left segment α is *less than* the left segment β, and that β is *greater than* α, and write $\beta > \alpha$.

Let α and β be left segments. Then $\alpha \subset \beta$ if and only if either $\alpha = \beta$ or $\alpha < \beta$. It follows that $\alpha \subset \beta$ (i.e., α is a subset of β) if and only if α is less than or equal to β, and therefore it is appropriate to indicate the inclusion relationship $\alpha \subset \beta$ between left segments α and β by either by writing $\alpha \leq \beta$ or else by writing $\beta \geq \alpha$.

The following proposition establishes that the ordering of left segments of the rational numbers defined as described above is *transitive*.

Proposition A.4 *Let α, β and γ be left segments. Suppose that $\alpha < \beta$ and $\beta < \gamma$. Then $\alpha < \gamma$.*

Proof The left segments α, β and γ satisfy $\alpha \subset \beta \subset \gamma$, because $\alpha < \beta$ and $\beta < \gamma$. If it were the case that $\alpha = \gamma$ then it would follow that both $\alpha = \beta$ and $\beta = \gamma$. But $\alpha \neq \beta$ and $\beta \neq \gamma$. Therefore $\alpha \neq \gamma$, and therefore $\alpha < \gamma$, as required. ∎

The following proposition establishes that the ordering of left segments of the rational numbers defined as described above satisfies the *Trichotomy Law*.

Proposition A.5 *Let α and β be left segments of the rational numbers. Then one and only one of the three statements $\alpha < \beta$, $\alpha = \beta$ and $\beta < \alpha$ is true.*

Proof Two sets are equal to one another if and only if they have the same elements. It follows that α and β are unequal if and only if there exists some rational number that belongs to one of the left segments α and β but not to the other. But it then follows from Lemma A.3 that either $\alpha \subset \beta$, in which case $\alpha < \beta$, or else $\beta \subset \alpha$, in which case $\beta < \alpha$. Now at most one of the properties $\alpha = \beta$, $\alpha < \beta$ and $\beta < \alpha$ can hold for the left segments α and β. The result follows. ∎

A.2 Addition and Subtraction of Left Segments

We now discuss how an appropriate operation of *addition* is to be defined on the set of left segments of the rational numbers. The definition of addition for such left segments should be compatible with the definition of addition for rational numbers. Therefore we investigate the relationship between the left segments λ_q, λ_r and λ_{q+r} representing rational numbers q, r and $q+r$ respectively.

Lemma A.6 *Let q and r be rational numbers, and let*

$$\lambda_q = \{s \in \mathbb{Q} : s < q\} \quad \lambda_r = \{t \in \mathbb{Q} : t < r\}$$

and

$$\lambda_{q+r} = \{t \in \mathbb{Q} : t < q + r\}.$$

Let p be a rational number. Then $p \in \lambda_{q+r}$ if and only if there exist rational numbers s and t such that $s \in \lambda_q$, $t \in \lambda_r$ and $s + t = p$.

Proof Let p be a rational number. First suppose that there exist rational numbers s and t for which $s \in \lambda_q$, $t \in \lambda_r$ and $s + t = p$. Then $s < q$, $t < r$, and therefore $p = s + t < q + r$. It follows that $p \in \lambda_{q+r}$.

Now let $p \in \lambda_{q+r}$. Then $p < q + r$. A rational number v can be chosen such that $p < v < q + r$. Let

$$s = p + q - v \quad \text{and} \quad t = v - q.$$

Then $s < q$ and $t < r$ and therefore $s \in \lambda_q$ and $t \in \lambda_r$. Moreover $s + t = p$. The result follows. ∎

Lemma A.7 *Let α and β be left segments of the rational numbers, and let*

$$\gamma = \{p \in \mathbb{Q} : \textit{ there exist } s \in \alpha \textit{ and } t \in \beta \textit{ for which } p = s + t\}.$$

Then γ is a left segment of the rational numbers.

Proof We must verify that the four properties listed in the definition of left segments are satisfied. First we note that α and β are non-empty, because they are left segments, and therefore γ is non-empty. Thus γ satisfies property (i) in the definition of left segments.

Property (ii) in the definition of left segments ensures that there exist rational numbers u and v for which $u \notin \alpha$ and $v \notin \beta$. It then follows from Lemma A.1 that $s < u$ for all $s \in \alpha$ and $t < v$ for all $t \in \beta$. But then $s + t < u + v$ for all $s \in \alpha$ and $t \in \beta$, and therefore $q < u + v$ for all $q \in \gamma$.

Therefore γ is not the entire set of rational numbers, and thus γ satisfies property (ii) in the definition of left segments.

Next let u and p be rational numbers where $p \in \gamma$ and $u < p$. Then there exist rational numbers s and t such that $s \in \alpha$, $t \in \beta$ and $s + t = p$. Let $v = s + u - p$. Then $v < s$. Property (iii) in the definition of left segments therefore ensures that $v \in \alpha$. Also $v + t = s + t + u - p = u$. It follows that $u \in \gamma$. Thus γ satisfies property (iii) in the definition of left segments.

Finally let q be a rational number belonging to γ. Then there exist $s \in \alpha$ and $t \in \beta$ such that $s+t = q$. Property (iv) in the definition of left segments then ensures that there exists some rational number u such that $u > s$ and $u \in \alpha$. Let $r = u + t$. Then $r > q$ and $r \in \gamma$. Thus γ satisfies property (iv) in the definition of left segments. This completes the proof. ∎

Definition Let α and β be left segments of the rational numbers. The *sum* $\alpha+\beta$ of the left segments α and β is the left segment of the rational numbers defined so that

$$\alpha + \beta = \{p \in \mathbb{Q} : \text{ there exist } s \in \alpha \text{ and } t \in \beta \text{ for which } p = s + t\}.$$

Lemma A.8 *Let q and r be rational numbers, and let*

$$\lambda_q = \{s \in \mathbb{Q} : s < q\}, \quad \lambda_r = \{t \in \mathbb{Q} : t < r\}$$

and

$$\lambda_{q+r} = \{p \in \mathbb{Q} : p < q + r\}.$$

Then $\lambda_q + \lambda_r = \lambda_{q+r}$.

Proof This identity follows immediately from Lemma A.6 and the definition of the sum $\lambda_q + \lambda_r$ of the left segments λ_q and λ_r. ∎

The following proposition notes that the operation of addition defined on the set of left segments of the rational numbers as described above is *commutative*.

Proposition A.9 *Let α and β be left segments of the rational numbers. Then*

$$\alpha + \beta = \beta + \alpha.$$

Proof This identity follows immediately from the definition of addition of left segments and the commutativity of addition in the field of rational numbers. ∎

The following proposition shows that the operation of addition defined on the set of left segments of the rational numbers as described above is *associative.*

Proposition A.10 *Let α, β and γ be left segments of the rational numbers. Then*

$$(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma) = \{p + q + r : p \in \alpha, \ \ q \in \beta \ \text{and} \ r \in \gamma\}.$$

Proof Throughout this proof, let

$$\theta = \{p + q + r : p \in \alpha, \ \ q \in \beta \text{ and } r \in \gamma\}.$$

Let $p \in \alpha$, $q \in \beta$ and $r \in \gamma$. Then $p + q \in \alpha + \beta$, and therefore

$$p + q + r \in (\alpha + \beta) + \gamma.$$

Similarly

$$p + q + r \in \alpha + (\beta + \gamma).$$

It follows that

$$\theta \subset (\alpha + \beta) + \gamma \quad \text{and} \quad \theta \subset \alpha + (\beta + \gamma).$$

Now let $s \in (\alpha + \beta) + \gamma$. Then there exist $t \in \alpha + \beta$ and $r \in \gamma$ such that $s = t + r$. But then there exist $p \in \alpha$ and $q \in \beta$ such that $t = p + q$. Then $s = p + q + r$ and thus $s \in \theta$. We conclude that

$$(\alpha + \beta) + \gamma \subset \theta.$$

Similarly

$$\alpha + (\beta + \gamma) \subset \theta.$$

When we combine these set inclusions with the inclusions in the reverse direction already established, we find that

$$(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma) = \theta,$$

as required. ∎

Proposition A.11 *Let q be a rational number, and let λ_q be the left segment representing q, so that*

$$\lambda_q = \{s \in \mathbb{Q} : s < q\}.$$

Then

$$\lambda_q + \alpha = \alpha + \lambda_q = \{p + q : p \in \alpha\}$$

for all left segments α of the rational numbers.

Proof Let α be a left segment of the rational numbers, and let

$$\gamma = \{s + q : s \in \alpha\}$$

Let $s \in \alpha$ and $t \in \lambda_q$. Then $t < q$, and $s + t = u + q$, where $u = s + t - q$. Moreover $u < s$. It therefore follows from property (iii) in the definition of left segments that $u \in \alpha$ and therefore $u + q \in \gamma$. But $u + q = s + t$. We conclude that $s + t \in \gamma$ for all $s \in \alpha$ and $t \in \lambda_q$, and therefore $\alpha + \lambda_q \subset \gamma$.

Now let $s \in \alpha$. It follows from property (iv) in the definition of left segments that there exists some rational number r such that $r \in \alpha$ and $r > s$. Let $t = s + q - r$. Then $t < q$ and $r + t = s + q$. But $r \in \alpha$ and $t \in \lambda_q$, and therefore $r + t \in \alpha + \lambda_q$. It follows that $s + q \in \alpha + \lambda_q$ for all $s \in \alpha$. We conclude that

$$\gamma \subset \alpha + \lambda_q \subset \gamma,$$

and therefore $\alpha + \lambda_q = \gamma$.

The operation of addition of left segments is commutative. It follows that $\lambda_q + \alpha = \alpha + \lambda_q = \gamma$, as required. ∎

The following corollary establishes that the set λ_0 of strictly negative rational numbers is a zero element for the operation of addition defined on the set of left segments of the rational numbers.

Corollary A.12 *Let λ_0 be the left segment representing zero, so that*

$$\lambda_0 = \{n \in \mathbb{Q} : n < \lambda_0\}.$$

Then

$$\lambda_0 + \alpha = \alpha + \lambda_0 = \alpha$$

for all left segments α of the rational numbers.

Proof This result follows directly from Proposition A.11. ∎

Lemma A.13 *Let q and r be rational numbers, and let*

$$\lambda_q = \{s \in \mathbb{Q} : s < q\} \quad \lambda_r = \{t \in \mathbb{Q} : t < r\}$$

and

$$\lambda_{q-r} = \{t \in \mathbb{Q} : t < q - r\}.$$

Let p be a rational number. Then $p \in \lambda_{q-r}$ if and only if there exists some rational number s such that $s > p$ and $s + t \in \lambda_q$ for all $t \in \lambda_r$.

Proof Let p be a rational number. Suppose that there exists some rational number s such that $s > p$ and $s + t \in \lambda_q$ for all $t \in \lambda_r$. Then $q - s$ cannot be less than r, because $s + (q - s) = q$ and $q \notin \lambda_q$. It follows that $r \leq q - s$, and therefore $s \leq q - r$. It follows that $p < q - r$, and therefore $p \in \lambda_{q-r}$.

Now suppose that $p \in \lambda_{q-r}$. Then $p < q-r$. A rational number s can then be found so that $p < s < q - r$. (For example one may take $s = \frac{1}{2}(p+q-r)$.) If t is any rational number satisfying $t < r$ then $s + t < q - r + t < q$. It follows that $s + t \in \lambda_q$ for all $t \in \lambda_r$. The result follows. ∎

Lemma A.13 establishes a relationship between the left segments λ_q, λ_r and λ_{q-r} representing q, r and $q - r$ for all rational numbers q and r. This relationship suggests the criterion that should be used in order to define a subset $\alpha - \beta$ of the rational numbers that is intended to represent the difference of two left segments α and β of the rational numbers. A rational number p should belong to this subset $\alpha - \beta$ if and only if there exists some rational number s with the properties that $s > p$ and $s + t \in \alpha$ for all $t \in \beta$. We prove below that the set $\alpha - \beta$ defined in this fashion is indeed a left segment of the rational numbers, and furthermore that it has the properties that it ought to have if it is to represent the difference of two left segments of the rational numbers.

Lemma A.14 *Let α and β be left segments and let $\alpha - \beta$ be the subset of the set of rational numbers defined so that a rational number p belongs to $\alpha - \beta$ if and only if there exists some rational number s with the properties that $s > p$ and $s + t \in \alpha$ for all $t \in \beta$. Then $\alpha - \beta$ is a left segment of the rational numbers.*

Proof Properties (i) and (ii) in the definition of left segments ensure that there exist rational numbers q, r, u and v such that $q \in \alpha$, $r \notin \alpha$, $u \in \beta$ and $v \notin \beta$. If $t \in \beta$ then $t < v$ (Lemma A.1) and therefore $q - v + t < q$. Property (iii) in the definition of left segments then ensures that $q - v + t \in \alpha$. Thus $q - v$ is a rational number with the property that $q - v + t \in \alpha$ for all $t \in \beta$. It therefore follows from the definition of $\alpha - \beta$ that all rational numbers less than $q - v$ belong to $\alpha - \beta$, and thus $\alpha - \beta \neq \emptyset$. Thus $\alpha - \beta$ satisfies property (i) in the definition of left segments.

Also $u \in \beta$ and $(r - u) + u \notin \alpha$, and therefore $r - u \notin \alpha - \beta$. We conclude that $\alpha - \beta$ is not the entire set of rational numbers, and thus $\alpha - \beta$ satisfies property (ii) in the definition of left segments.

The definition of $\alpha - \beta$ ensures that this subset of the set of rational numbers satisfies properties (iii) and (iv) in the definition of left segments. We conclude that $\alpha - \beta$ is a left segment of the rational numbers, as required. ∎

Definition Let α and β be left segments. The *difference* $\alpha - \beta$ of β is defined to be the left segment of the rational numbers characterized by the following property:

> A rational number p belongs to $\alpha - \beta$ if and only if there exists some rational number s with the properties that $s > p$ and $s + t \in \alpha$ for all $t \in \beta$.

Lemma A.15 *Let q and r be rational numbers, and let*

$$\lambda_q = \{s \in \mathbb{Q} : s < q\}, \quad \lambda_r = \{t \in \mathbb{Q} : t < r\}$$

and

$$\lambda_{q-r} = \{p \in \mathbb{Q} : p < q - r\}.$$

Then $\lambda_q - \lambda_r = \lambda_{q-r}$.

Proof This identity follows immediately from Lemma A.13 and the definition of the difference $\lambda_q - \lambda_r$ of the left segments λ_q and λ_r. ∎

Remark Let q and r be rational numbers, and let

$$\lambda_q = \{s \in \mathbb{Q} : s < q\} \text{ and } \lambda_r = \{t \in \mathbb{Q} : t < r\}.$$

Then

$$\{p \in \mathbb{Q} : p + t \in \lambda_q \text{ for all } t \in \lambda_r\} = \{p \in \mathbb{Q} : p \leq q - r\}.$$

Now the set $\{p \in \mathbb{Q} : p \leq q - r\}$ does not satisfy property (iv) in the definition of left segments of the rational numbers. It follows that care needs to be exercised in formulating an appropriate definition for the operation of subtraction of left segments of the rational numbers. We could not simply define the difference of left segments α and β to be the set of rational numbers p with the property that $p + t \in \alpha$ for all $t \in \beta$, because the resulting set would not a left segment. It is for this reason that we define $\alpha - \beta$ to be the set of all rational numbers p for which there exists some other rational number s with the properties that $s > p$ and $s + t \in \alpha$ for all $t \in \beta$.

Proposition A.16 *Let α and β be a left segment of the rational numbers. Then $(\alpha - \beta) + \beta = \alpha$.*

Proof It follows from the definition of $\alpha - \beta$ that $u + v \in \alpha$ for all $u \in \alpha - \beta$ and $v \in \beta$. Thus $(\alpha - \beta) + \beta \subset \alpha$.

Let p be a rational number belonging to the left segment α. Two successive applications of property (iv) in the definition of left segments of the natural numbers ensure the existence of rational numbers p_1 and p_2 such that $p < p_1 < p_2$ and $p_2 \in \alpha$. It then follows from Lemma A.2 that there exist rational numbers v and w such that $v \in \beta$, $w \notin \beta$ and $w - v < p_2 - p_1$. Let $u = p - v$ and $s = p_1 - v$. Then $u < s$. Moreover if t is a rational number belonging to β then $t < w$, because $w \notin \beta$ (Lemma A.1), and therefore

$$s + t = p_1 - v + t < p_1 - v + w < p_2,$$

and therefore $s + t \in \alpha$. Thus $u < s$, where $s + t \in \alpha$ for all $t \in \beta$. It follows from the definition of $\alpha - \beta$ that $u \in \alpha - \beta$. Thus $p = u + v$, where $u \in \alpha - \beta$ and $v \in \beta$, and therefore $p \in (\alpha - \beta) + \beta$. We conclude therefore that

$$\alpha \subset (\alpha - \beta) + \beta \subset \alpha,$$

and therefore

$$\alpha = (\alpha - \beta) + \beta,$$

as required. ∎

Definition Let α be left segments. The *negative* $-\alpha$ of α is the left segment of the rational numbers defined so that $\alpha = \lambda_0 - \alpha$, where λ_0, denotes the zero left segment consisting of all strictly negative rational numbers. Accordingly a rational number p belongs to $-\alpha$ if and only if there exists some rational number s with the properties that $s > p$ and $s + t < 0$ for all $t \in \alpha$.

Corollary A.17 *Let α be a left segment of the rational numbers, and let $-\alpha$ denote the negative of α. Then*

$$\alpha + (-\alpha) = (-\alpha) + \alpha = \lambda_0$$

where $\lambda_0 = \{q \in \mathbb{Q} : q < 0\}$.

Proof Applying Proposition A.16, and making use of the commutativity of the operation of addition defined on the set of left segments of the rational numbers, we see that

$$\alpha + (-\alpha) = (-\alpha) + \alpha = (\lambda_0 - \alpha) + \alpha = \lambda_0,$$

as required. ∎

Corollary A.18 *Let α and β be left segments of the rational numbers. Then $\alpha = -\beta$ if and only if $\alpha + \beta = 0$. Also $\beta = -\alpha$ if and only if $\alpha + \beta = 0$.*

Proof It follows from Corollary A.17 that if $\alpha = -\beta$ then $\alpha+\beta = (-\beta)+\beta = \lambda_0$. Similarly if $\beta = -\alpha$ then $\alpha + \beta = \alpha + (-\alpha) = \lambda_0$. On the other hand, by making use of the associativity of the operation of addition defined on the set of left segments of the rational numbers together with Corollaries A.12 and A.17, we see that if $\alpha + \beta = \lambda_0$ then

$$\alpha = \alpha + \lambda_0 = \alpha + (\beta + (-\beta)) = (\alpha + \beta) + (-\beta) = \lambda_0 + (-\beta) = -\beta$$

and

$$\beta = \lambda_0 + \beta = ((-\alpha) + \alpha) + \beta = (-\alpha) + (\alpha + \beta) = (-\alpha) + \lambda_0 = -\alpha,$$

as required. ∎

Corollary A.19 *Let α be a left segment of the rational numbers. Then*

$$-(-\alpha) = \alpha.$$

Proof It follows from Corollary A.18 that if $\beta = -\alpha$ then $\alpha = -\beta = -(-\alpha)$, as required. ∎

Many of the results already obtained that concern the operation of addition defined on the set of left segments of the rational numbers are collected and summarized in the following proposition.

Proposition A.20 *An operation of addition may be defined on the set of all left segments of the rational numbers characterized by the following condition:*

- *a rational number p belongs to the sum $\alpha + \beta$ of left segments α and β if and only if there exist rational numbers s and t such that $s \in \alpha$, $t \in \beta$ and $p = s + t$.*

This operation of addition defined on the set of left segments of the rational numbers satisfies the following properties:—

- (commutativity of addition) $\alpha + \beta = \beta + \alpha$ *for all left segments α and β;*
- (associativity of addition) $(\alpha+\beta)+\gamma = \alpha+(\beta+\gamma)$ *for all left segments α, β and γ;*
- (zero element) $\alpha + \lambda_0 = \lambda_0 + \alpha$ *for all left segments α, where*

$$\lambda_0 = \{q \in \mathbb{Q} : q < 0\}.$$

- (existence of negatives) *given any left segment α of the rational numbers, there exists a left segment $-\alpha$, the negative of α, characterized by the property that $\alpha + (-\alpha) = (-\alpha) + \alpha = \lambda_0$.*

The set consisting of all left segments of the rational numbers is then a commutative group with respect to the operation of addition defined on left segments.

Proof This result is established by combining the results of Proposition A.9, Proposition A.10, Corollary A.12, and Corollary A.17. ∎

The following proposition shows that the operation of addition defined on the set of left segments of the rational numbers as described above satisfies an appropriate cancellation law.

Proposition A.21 *Let α, β and γ be left segments. Suppose that $\alpha + \gamma = \beta + \gamma$. Then $\alpha = \beta$.*

Proof The equality $\alpha + \gamma = \beta + \gamma$ is satisfied, and therefore

$$\begin{aligned} \alpha &= \alpha + \lambda_0 = \alpha + (\gamma + (-\gamma)) \\ &= (\alpha + \gamma) + (-\gamma) = (\beta + \gamma) + (-\gamma) \\ &= \beta + (\gamma + (-\gamma)) = \beta + \lambda_0 \\ &= \beta, \end{aligned}$$

as required. ∎

Lemma A.22 *Let α and β be left segments of the rational numbers. Then*

$$\alpha - \beta = \alpha + (-\beta).$$

Proof Applying the properties of addition of left segments summarized in Proposition A.20, together with the identity $-\beta = \lambda_0 - \beta$ that characterizes $-\beta$ and the result of Proposition A.16, we find that

$$\begin{aligned} (\alpha - \beta) + \beta &= \alpha = \alpha + \lambda_0 = \alpha + ((\lambda_0 - \beta) + \beta) \\ &= \alpha + ((-\beta) + \beta) = (\alpha + (-\beta)) + \beta. \end{aligned}$$

It then follows from the cancellation property of addition Proposition A.21 that $\alpha - \beta = \alpha + (-\beta)$, as required. ∎

Lemma A.23 *Let α and β be left segments of the rational numbers. Then $\beta - \alpha = -(\alpha - \beta)$.*

Proof Applying Lemma A.22, and the properties of addition of left segments summarized in Proposition A.20, we find that

$$\begin{aligned}(\alpha - \beta) + (\beta - \alpha) &= (\alpha + (-\beta)) + (\beta + (-\alpha)) \\ &= ((\alpha + (-\beta)) + \beta) + (-\alpha) \\ &= (\alpha + ((-\beta) + \beta)) + (-\alpha) \\ &= (\alpha + \lambda_0) + (-\alpha) \\ &= \alpha + (-\alpha) = \lambda_0.\end{aligned}$$

It then follows from Corollary A.18 that $\beta - \alpha = -(\alpha - \beta)$, as required. ∎

Alternative Proof Applying Proposition A.16 and the associativity of the operation of addition on left segments, we find that

$$\begin{aligned}((\alpha - \beta) + (\beta - \alpha)) + \alpha &= (\alpha - \beta) + ((\beta - \alpha) + \alpha) \\ &= (\alpha - \beta) + \beta = \alpha \\ &= \lambda_0 + \alpha.\end{aligned}$$

Applying the cancellation law expressed in Proposition A.21, it follows that

$$(\alpha - \beta) + (\beta - \alpha)) = \lambda_0.$$

It then follows from Corollary A.18 that $\beta - \alpha = -(\alpha - \beta)$, as required. ∎

Lemma A.24 *Let α and β be left segments of the rational numbers. Then*

$$(-\alpha) + (-\beta) = -(\alpha + \beta).$$

Proof Applying the properties of addition of left segments summarized in Proposition A.20, we find that

$$\begin{aligned}((-\alpha) + (-\beta)) + (\beta + \alpha) &= (-\alpha) + ((-\beta) + (\beta + \alpha)) \\ &= (-\alpha) + (((-\beta) + \beta) + \alpha) \\ &= (-\alpha) + (\lambda_0 + \alpha) = (-\alpha) + \alpha \\ &= \lambda_0.\end{aligned}$$

It follows from Corollary A.18 that

$$(-\alpha) + (-\beta) = -(\beta + \alpha).$$

But the operation of addition defined on left segments of the rational numbers is commutative (Proposition A.9). Therefore $\alpha + \beta = \beta + \alpha$, and thus

$$(-\alpha) + (-\beta) = -(\alpha + \beta),$$

as required.

Proposition A.25 *Let α, β and γ be left segments of the natural numbers. Suppose that $\alpha < \beta$. Then $\alpha + \gamma < \beta + \gamma$.*

Proof It follows from the inequality $\alpha < \beta$ that $\alpha \subset \beta$. It then follows from the definition of addition of left segments that $\alpha + \gamma \subset \beta + \gamma$. Therefore either $\alpha + \gamma = \beta + \gamma$ or $\alpha + \gamma < \beta + \gamma$.

Now if it were the case that $\alpha + \gamma = \beta + \gamma$ it would follow from Proposition A.21 that $\alpha = \beta$. But $\alpha \neq \beta$. Therefore $\alpha + \gamma \neq \beta + \gamma$, and hence $\alpha + \gamma < \beta + \gamma$, as required.

We now discuss the definition and basic properties of the operation of multiplication defined on left segments of the rational numbers. Given any left segment α of the rational numbers, we define

$$\alpha^+ = \{q \in \alpha : q > 0\}.$$

We refer to α^+ as the *positive part* of the left segment α.

Definition A left segment α of the rational numbers is said to be *positive* if $\alpha > \lambda_0$, where $\lambda_0 = \{q \in \mathbb{Q} : q < 0\}$.

Lemma A.26 *Let α be a left segment of the rational numbers, and let $\lambda_0 = \{q \in \mathbb{Q} : q < 0\}$. Then α is positive if and only if $\alpha^+ \neq \emptyset$.*

Proof Suppose that α is positive. Then $\alpha > \lambda_0$, and therefore $\lambda_0 \subset \alpha$. But $\lambda_0 \neq \alpha$. It follows that there exists a rational number q such that $q \in \alpha$ but $q \notin \lambda_0$. The definition of λ_0 then ensures that $q \geq 0$. Property (iv) in the definition of left segments then ensures that there exists $r \in \alpha$ satisfying $r > q$. Then $r > 0$, and therefore $r \in \alpha^+$. Thus $\alpha^+ \neq \emptyset$.

Conversely suppose that $\alpha^+ \neq \emptyset$. Let $r \in \alpha^+$. Then $r > 0$, and therefore $r \notin \lambda_0$. Moreover $q < r$ for all $q \in \lambda_0$. It follows from property (iii) in the definition of left segments that $q \in \alpha$ for all $q \in \lambda_0$, and thus $\lambda_0 \subset \alpha$. But $\lambda_0 \neq \alpha$, because $r \in \alpha$ and $r \notin \lambda_0$. Therefore $\alpha > \lambda_0$. This completes the proof. ∎

Lemma A.27 *The sum of two positive left segments of the rational numbers is itself positive.*

Proof Let α and β be positive left segments of the rational numbers. Then $\alpha^+ \neq \emptyset$ and $\beta^+ \neq \emptyset$. Let $s \in \alpha^+$ and $t \in \beta^+$. Then $s + t \in \alpha + \beta$ and $s + t > 0$, and therefore the left segment $\alpha + \beta$ is also positive. The result follows. ∎

Definition A left segment α of the rational numbers is said to be *zero* if $\alpha = \lambda_0$, where $\lambda_0 = \{q \in \mathbb{Q} : q < 0\}$.

Definition A left segment α of the rational numbers is said to be *negative* if $-\alpha$ is positive.

Lemma A.28 *Every left segment of the rational numbers is positive, negative or zero. A left segment cannot be both zero and positive, both zero and negative, or both positive and negative.*

Proof Let α be a left segment of the rational numbers. It follows from Proposition A.25 that if $\alpha < \lambda_0$ then

$$\lambda_0 = \alpha + (-\alpha) < \lambda_0 + (-\alpha) < -\alpha,$$

and thus $-\alpha$ is positive. Conversely if $-\alpha$ is positive than $\lambda_0 < -\alpha$ and therefore

$$\alpha = \lambda_0 + \alpha < (-\alpha) + \alpha = \lambda_0.$$

Thus the left segment α is negative if and only if $\alpha < \lambda_0$. The required result therefore follows from Proposition A.5. ∎

A.3 Multiplication and Division of Positive Left Segments

Lemma A.29 *Let q and r be positive rational numbers, and let*

$$\lambda_q = \{s \in \mathbb{Q} : s < q\} \quad \lambda_r = \{t \in \mathbb{Q} : t < r\}$$

and

$$\lambda_{qr} = \{t \in \mathbb{Q} : t < qr\}.$$

Let p be a rational number. Then $p \in \lambda_{qr}$ if and only if there exist rational numbers s and t such that $s \in \lambda_q$, $t \in \lambda_r$, $s > 0$, $t > 0$ and $p \leq st$.

Proof Let p be a rational number. Suppose that there exist rational numbers s and t such that $s \in \lambda_q$, $t \in \lambda_r$, $s > 0$, $t > 0$ and $p \leq st$. Then $p < qr$, because $s < q$ and $t < r$, and therefore $p \in \lambda_{qr}$.

Now let $p \in \lambda_{qr}$. Then $p < qr$. Rational numbers u and v can be chosen such that $u > 0$, $v > 0$ and $p < u < v < qr$. Let

$$s = \frac{uq}{v} \quad \text{and} \quad t = \frac{v}{q}.$$

Then $0 < s < q$, $0 < t < r$ and therefore $s \in \lambda_q$ and $t \in \lambda_r$. Moreover $st = u$, and therefore $p < st$. The result follows. ∎

Lemma A.29 establishes a relationship between the left segments λ_q, λ_r and λ_{qr} representing q, r and qr for all positive rational numbers q and r. This relationship suggests the criterion that should be used in order to define a subset $\alpha\cdot\beta$ of the rational numbers that is intended to represent the product of two positive left segments α and β of the rational numbers. A rational number p should belong to this subset $\alpha\cdot\beta$ if and only if there exist rational numbers s and t such that $s \in \alpha$, $t \in \beta$, $s > 0$, $t > 0$ and $p \leq st$. We prove below that the set $\alpha\cdot\beta$ defined in this fashion is indeed a left segment of the rational numbers, and furthermore that it has the properties that it ought to have if it is to represent the product of two positive left segments of the rational numbers.

Proposition A.30 *Let α and β be positive left segments of the rational numbers, and let $\alpha\cdot\beta$ be the subset of the rational numbers consisting of those rational numbers p that satisfy $p \leq st$ for some $s \in \alpha^+$ and $t \in \beta^+$ Then $\alpha\cdot\beta$ is itself a positive left segment.*

Proof The positive parts α^+ and β^+ of the left segments α and β respectively are non-empty, because α and β are positive left segments, and therefore $\alpha\cdot\beta$ is non-empty, and thus satisfies property (i) in the definition of left segments.

Also there exist positive rational numbers u and v such that $u \notin \alpha$ and $v \notin \beta$. Then $s < u$ for all $s \in \alpha^+$ and $t < v$ for all $t \in \beta_+$. Then $st < uv$ for all $s \in \alpha^+$ and $t \in \beta^+$, and therefore $q < uv$ for all $q \in \alpha\cdot\beta$. Therefore $\alpha\cdot\beta$ is not the entire set of rational numbers, and thus $\alpha\cdot\beta$ satisfies property (ii) in the definition of left segments.

It follows from the definition of $\alpha\cdot\beta$ that if $q \in \alpha\cdot\beta$ then $p \in \alpha\cdot\beta$ for all rational numbers p satisfying $p < q$. Thus $\alpha\cdot\beta$ satisfies property (iii) in the definition of left segments.

Let $q \in \alpha\cdot\beta$. Then there exist $s \in \alpha^+$ and $t \in \beta^+$ such that $q \leq st$. There then exist $u \in \alpha^+$ and $v \in \beta^+$ such that $s < u$ and $t < v$. Then $uv \in \alpha\cdot\beta$ and $q < uv$. Thus $\alpha\cdot\beta$ satisfies property (iv) in the definition of left segments. We now conclude that $\alpha\cdot\beta$ is a left segment. Moreover $st \in \alpha\cdot\beta$ and $st > 0$ for all $s \in \alpha^+$ and $t \in \beta^+$. It follows that $\alpha\cdot\beta$ is a positive left segment, as required. ∎

Definition Let α and β be positive left segments of the rational numbers satisfying $\alpha > \lambda_0$ and $\beta > \lambda_0$. The *product* $\alpha\cdot\beta$ of the left segments α and β is defined to be the positive left segment of the rational numbers characterized by the following property:

> A rational number p belongs to $\alpha\cdot\beta$ if and only if there exist positive rational numbers s and t such that $s \in \alpha$, $t \in \beta$ and $p \leq st$.

Lemma A.31 *Let q and r be positive rational numbers, and let*

$$\lambda_q = \{s \in \mathbb{Q} : s < q\}, \quad \lambda_r = \{t \in \mathbb{Q} : t < r\}$$

and

$$\lambda_{qr} = \{p \in \mathbb{Q} : p < qr\}.$$

Then $\lambda_q \cdot \lambda_r = \lambda_{qr}$.

Proof This identity follows immediately from Lemma A.29 and the definition of the product $\lambda_q \cdot \lambda_r$ of the left segments λ_q and λ_r. ∎

The following proposition notes that the operation of multiplication defined on the set of positive left segments of the rational numbers as described above is *commutative*.

Proposition A.32 *Let α, β be positive left segments of the rational numbers. Then*

$$\alpha \cdot \beta = \beta \cdot \alpha.$$

Proof This identity follows immediately from the definition of multiplication of positive left segments and the commutativity of multiplication in the field of rational numbers. ∎

The following proposition shows that the operation of multiplication defined on the set of positive left segments of the rational numbers as described above is *associative*.

Proposition A.33 *Let α, β and γ be positive left segments of the rational numbers. Then*

$$(\alpha \cdot \beta) \cdot \gamma = \alpha \cdot (\beta \cdot \gamma) = \theta,$$

where θ denotes the set consisting of those rational numbers s for which there exist $p \in \alpha^+$, $q \in \beta^+$ and $r \in \gamma^+$ satisfying $s \leq pqr$.

Proof Let $s \in \theta$. Then there exist $p \in \alpha^+$, $q \in \beta^+$ and $r \in \gamma^+$ for which $s \leq pqr$. Then $pq \in \alpha \cdot \beta$, and therefore

$$pqr \in (\alpha \cdot \beta) \cdot \gamma.$$

Similarly

$$pqr \in \alpha \cdot (\beta \cdot \gamma).$$

But $s \leq pqr$, and $(\alpha \cdot \beta) \cdot \gamma$ and $\alpha \cdot (\beta \cdot \gamma)$ are left segments of the rational numbers. It follows that the rational number s belongs to both $(\alpha \cdot \beta) \cdot \gamma$

and $\alpha \cdot (\beta \cdot \gamma)$. Thus every rational number belonging to θ belongs to both $(\alpha \cdot \beta) \cdot \gamma$ and $\alpha \cdot (\beta \cdot \gamma)$, and therefore

$$\theta \subset (\alpha \cdot \beta) \cdot \gamma \quad \text{and} \quad \theta \subset \alpha \cdot (\beta \cdot \gamma).$$

Now let $s \in (\alpha \cdot \beta) \cdot \gamma$. Then there exist $t \in \alpha \cdot \beta$ and $r \in \gamma$ such that $t > 0$, $r > 0$ and $s \leq tr$. But then there exist $p \in \alpha$ and $q \in \beta$ such that $p > 0$, $q > 0$ and $t \leq pq$. Then $s \leq pqr$ and thus $s \in \theta$. We conclude that

$$(\alpha \cdot \beta) \cdot \gamma \subset \theta.$$

Similarly

$$\alpha \cdot (\beta \cdot \gamma) \subset \theta.$$

When we combine these set inclusions with the inclusions in the reverse direction already established, we find that

$$(\alpha \cdot \beta) \cdot \gamma = \alpha \cdot (\beta \cdot \gamma) = \theta,$$

as required. ∎

Proposition A.34 *Let q be a positive rational number, and let λ_q be the left segment representing q, so that*

$$\lambda_q = \{s \in \mathbb{Q} : s < q\}.$$

Then $\alpha \cdot \lambda_q = \{pq : p \in \alpha\}$ for all positive left segments α.

Proof Let α be a positive left segment, and let

$$\beta = \{pq : p \in \alpha\}.$$

Then $\alpha^+ \neq \emptyset$ (Lemma A.26). Let $p \in \alpha^+$ and $s \in \lambda_q^+$. Then $p > 0$ and $0 < s < q$, and therefore $ps = tq$, where $t = pqs^{-1}$. Moreover $t < p$, and therefore $t \in \alpha$, because α satisfies property (iii) in the definition of left segments. It follows that $ps \in \beta$. We conclude therefore that $\alpha \cdot \lambda_q \subset \beta$.

Now let $p \in \alpha^+$. It follows from property (iv) in the definition of left segments that there exists some rational number r such that $r \in \alpha$ and $0 < p < r$. Let $s = pqr^{-1}$. Then $0 < s < q$ and $rs = pq$. But $rs \in \alpha \cdot \lambda_q$. Thus $pq \in \alpha \cdot \lambda_q$ for all $p \in \alpha$ satisfying $p > 0$. Also $pq \in \alpha \cdot \lambda_q$ for all non-positive rational numbers p. It follows that $pq \in \alpha \cdot \lambda_q$ for all $p \in \alpha$, and therefore $\beta \subset \alpha \cdot \lambda_q$. We have already proved that $\alpha \cdot \lambda_q \subset \beta$. We conclude therefore that $\alpha \cdot \lambda_q = \beta$, as required. ∎

Corollary A.35 *Let λ_1 be the left segment representing the number 1, so that*

$$\lambda_1 = \{n \in \mathbb{Q} : n < 1\}.$$

Then

$$\lambda_1 \cdot \alpha = \alpha \cdot \lambda_1 = \alpha$$

for all positive left segments α of the rational numbers.

Lemma A.36 *Let q and r be positive rational numbers, and let*

$$\lambda_q = \{s \in \mathbb{Q} : s < q\} \quad \lambda_r = \{t \in \mathbb{Q} : t < r\}$$

and

$$\lambda_{qr^{-1}} = \{t \in \mathbb{Q} : t < qr^{-1}\}.$$

Let p be a rational number. Then $p \in \lambda_{qr^{-1}}$ if and only if there exists some positive rational number s such that $s > p$ and $st \in \lambda_q$ for all $t \in \lambda_r$.

Proof Let p be a rational number. Suppose that there exists some positive rational number s such that $s > p$ and $st \in \lambda_q$ for all $t \in \lambda_r$. Then qs^{-1} cannot be less than r, because $s(qs^{-1}) = q$ and $q \notin \lambda_q$. It follows that $r \leq qs^{-1}$, and therefore $s \leq qr^{-1}$. It follows that $p < qr^{-1}$, and therefore $p \in \lambda_{qr^{-1}}$.

Now suppose that $p \in \lambda_{qr^{-1}}$. Then $p < qr^{-1}$. A positive rational number s can then be found so that $p < s < qr^{-1}$. If t is any rational number satisfying $t < r$ then $st < qr^{-1}t < q$. It follows that $st \in \lambda_q$ for all $t \in \lambda_r$. The result follows. ∎

Lemma A.36 establishes a relationship between the left segments λ_q, λ_r and $\lambda_{qr^{-1}}$ representing q, r and $q - r$ for all rational numbers q and r. This relationship suggests the criterion that should be used in order to define a subset α/β of the rational numbers that is intended to represent the difference of two positive left segments α and β of the rational numbers. A rational number p should belong to this subset α/β if and only if there exists some positive rational number s with the properties that $s > p$ and $st \in \alpha$ for all $t \in \beta$. We prove below that the set α/β defined in this fashion is indeed a left segment of the rational numbers, and furthermore that it has the properties that it ought to have if it is to represent the difference of two left segments of the rational numbers.

Lemma A.37 *Let α and β be positive left segments and let α/β be the subset of the set of rational numbers defined so that a rational number p belongs to α/β if and only if there exists some positive rational number s with the properties that $s > p$ and $st \in \alpha$ for all $t \in \beta$. Then α/β is a positive left segment of the rational numbers.*

Proof Properties (i) and (ii) in the definition of left segments ensure that there exist positive rational numbers q, r, u and v such that $q \in \alpha$, $r \notin \alpha$, $u \in \beta$ and $v \notin \beta$. If $t \in \beta$ then $t < v$ (Lemma A.1) and therefore $qv^{-1}t < q$. Property (iii) in the definition of left segments then ensures that $qv^{-1}t \in \alpha$. Thus qv^{-1} is a rational number with the property that $qv^{-1}t \in \alpha$ for all $t \in \beta$. It therefore follows from the definition of α/β that all rational numbers less than qv^{-1} belong to α/β, and thus $\alpha/\beta \neq \emptyset$. Thus α/β satisfies property (i) in the definition of left segments. Moreover this set α/β contains the positive rational number qv^{-1}.

Also $u \in \beta$ and $(ru^{-1})u \notin \alpha$, and therefore $ru^{-1} \notin \alpha/\beta$. We conclude that α/β is not the entire set of rational numbers, and thus α/β satisfies property (ii) in the definition of left segments.

The definition of α/β ensures that this subset of the set of rational numbers satisfies properties (iii) and (iv) in the definition of left segments. We conclude that α/β is a left segment of the rational numbers. Moreover this left segment includes a positive rational number and is therefore positive, as required. ∎

Definition Let α and β be positive left segments. The *quotient* α^{-1} of α is defined to be the positive left segment of the rational numbers characterized by the following property:

> A rational number p belongs to α/β if and only if there exists some positive rational number s with the properties that $s > q$ and $st \in \alpha$ for all $t \in \beta$.

Lemma A.38 *Let α be a positive left segment of the rational numbers. Then, given any rational number g satisfying $g > 1$, there exist rational numbers v and w such that $v \in \alpha$, $w \notin \alpha$ and $1 < wv^{-1} < g$.*

Proof There exists some rational number p for which $p > 0$ and $p \in \alpha$. It then follows from Lemma A.2 that there exist rational numbers v and w such that $v \in \alpha$, $w \notin \alpha$, $v < w$ and $w - v < p(g-1)$. Moreover we may choose v so that $v > p$. Then

$$0 < \frac{w}{v} - 1 = \frac{w-v}{v} < \frac{w-v}{p} < g - 1,$$

and therefore $1 < wv^{-1} < g$, as required. ∎

Proposition A.39 *Let α and β be positive left segments of the rational numbers. Then $(\alpha/\beta) \cdot \beta = \alpha$.*

Proof Let $p \in \alpha/\beta$, where $p > 0$. Then $pt \in \alpha$ for all $t \in \beta$. It follows from the definition of multiplication of positive left segments that $(\alpha/\beta) \cdot \beta \subset \alpha$.

Let p be a positive rational number belonging to the left segment α. Two successive applications of property (iv) in the definition of left segments of the natural numbers ensure the existence of rational numbers p_1 and p_2 such that $p < p_1 < p_2$ and $p_2 \in \alpha$. It then follows from Lemma A.38 that there exist positive rational numbers v and w such that $v \in \beta$, $w \notin \beta$ and $wv^{-1} < p_2 p_1^{-1}$. Let $u = pv^{-1}$ and $s = p_1 v^{-1}$. Then $u < s$. Moreover if t is a rational number belonging to β then $t < w$, because $w \notin \beta$ (Lemma A.1), and therefore

$$st = p_1 v^{-1} t < p_1 v^{-1} w < p_2,$$

and therefore $st \in \alpha$. Thus $u < s$, where $s > 0$ and $st \in \alpha$ for all $t \in \beta$. It follows from the definition of α/β that $u \in \alpha/\beta$. Thus $p = uv$, where u and v are positive rational numbers, $u \in \alpha/\beta$ and $v \in \beta$, and therefore $p \in (\alpha/\beta) \cdot \beta$. We conclude therefore that

$$\alpha \subset (\alpha/\beta) \cdot \beta \subset \alpha,$$

and therefore

$$\alpha = (\alpha/\beta) \cdot \beta,$$

as required. ∎

Definition Let α be a positive left segment. The *reciprocal* α^{-1} of α is defined by the identity $\alpha^{-1} = \lambda_1/\alpha$, where $\lambda_1 = \{q \in \mathbb{Q} : q < 1\}$.

Corollary A.40 *Let α be a positive left segment of the rational numbers, and let $alpha^{-1}$ denote the reciprocal of α. Then*

$$\alpha \cdot \alpha^{-1} = \alpha^{-1} \cdot \alpha = \lambda_1$$

where $\lambda_1 = \{q \in \mathbb{Q} : q < 1\}$.

Proof Applying Proposition A.39, and making use of the commutativity of the operation of multiplication defined on the set of positive left segments of the rational numbers, we see that

$$\alpha \cdot \alpha^{-1} = \alpha^{-1} \cdot \alpha = (\lambda_1/\alpha) \cdot \alpha = \lambda_1,$$

as required. ∎

The following proposition shows that the operation of multiplication defined on the set of positive left segments of the rational numbers as described above is *distributive* over the operation of addition.

Proposition A.41 *Let α, β and γ be positive left segments of the rational numbers. Then $(\alpha + \beta) \cdot \gamma = \alpha \cdot \gamma + \beta \cdot \gamma$.*

Proof Let $q \in (\alpha + \beta) \cdot \gamma$. Then there exist $p \in \alpha + \beta$ and $q_3 \in \gamma$ such that $p > 0$, $q_3 > 0$ and $q \leq pq_3$. It then follows from the definition of addition of left segments that there exist $p_1 \in \alpha$ and $p_2 \in \beta$ such that $p = p_1 + p_2$. There then exist $q_1 \in \alpha$ and $q_2 \in \beta$ such that $q_1 > 0$, $q_2 > 0$, $q_1 \geq p_1$ and $q_2 \geq p_2$, because α and β are positive left segments. Then $0 < p = p_1 + p_2 \leq q_1 + q_2$. It follows that $q \leq q_1 q_3 + q_2 q_3$. But $q_1 q_3 \in \alpha \cdot \gamma$, $q_2 q_3 \in \beta \cdot \gamma$, and therefore $q_1 q_3 + q_2 q_3 \in \alpha \cdot \gamma + \beta \cdot \gamma$. Also $\alpha \cdot \gamma + \beta \cdot \gamma$ is a left segment of the rational numbers. It follows that $q \in \alpha \cdot \gamma + \beta \cdot \gamma$. We conclude therefore that

$$(\alpha + \beta) \cdot \gamma \subset \alpha \cdot \gamma + \beta \cdot \gamma.$$

Now let $q \in \alpha \cdot \gamma + \beta \cdot \gamma$. Then there exist $p_1 \in \alpha \cdot \gamma$ and $p_2 \in \beta \cdot \gamma$ such that $q = p_1 + p_2$ There then exist $q_1 \in \alpha$, $q_2 \in \beta$ and $q_3 \in \gamma$ and $q_4 \in \gamma$ such that $q_1 > 0$, $q_2 > 0$, $q_3 > 0$ and $q_4 > 0$ $p_1 \leq q_1 q_3$ and $p_2 \leq q_2 q_4$. Let q_5 be the maximum of q_3 and q_4. Then $p_1 \leq q_1 q_5$ and $p_2 \leq q_2 p_5$, and therefore $q \leq (q_1 + q_2) p_5$. Moreover $(q_1 + q_2) q_5 \in (\alpha + \beta) \cdot \gamma$. We conclude therefore that

$$\alpha \cdot \gamma + \beta \cdot \gamma \subset (\alpha + \beta) \cdot \gamma.$$

The set inclusion in the other direction has already been verified. Therefore

$$(\alpha + \beta) \cdot \gamma = \alpha \cdot \gamma + \beta \cdot \gamma,$$

as required. ∎

Proposition A.42 *Let α, β and γ be positive left segments of the rational numbers. Suppose that $\alpha - \beta$ is positive. Then $(\alpha - \beta) \cdot \gamma = \alpha \cdot \gamma - \beta \cdot \gamma$.*

Proof It follows from Proposition A.16 that $(\alpha - \beta) + \beta = \alpha$. Moreover $\alpha - \beta$ and β are both positive. It therefore follows from Proposition A.41 that

$$(\alpha - \beta) \cdot \gamma + \beta \cdot \gamma = \alpha \cdot \gamma.$$

Adding $-(\beta \cdot \gamma)$ to both sides, we find that

$$(\alpha - \beta) \cdot \gamma = \alpha \cdot \gamma - \beta \cdot \gamma,$$

as required. ∎

A.4 Multiplication and Division of Left Segments of Arbitrary Sign

Definition The operation of multiplication is extended from the set of positive left segments of the rational numbers to the set of all left segments of the rational numbers (positive, negative or zero), so as to satisfy the following rules:

- $\alpha \cdot \beta = \lambda_0$ if either $\alpha = \lambda_0$ or $\beta = \lambda_0$.
- $(-\rho) \cdot \sigma = \rho \cdot (-\sigma) = -(\rho \cdot \sigma)$ and $(-\rho) \cdot (-\sigma) = \rho \cdot \sigma$ for all positive left segments ρ and σ of the rational numbers.

Proposition A.43 *Let α and β be left segments of the rational numbers. Then*

$$\alpha \cdot \beta = \beta \cdot \alpha.$$

Proof Let ρ and σ be positive left segments of the rational numbers. Proposition A.32 ensures that $\rho \cdot \sigma = \sigma \cdot \rho$. Then

$$\begin{array}{rcl} (-\rho) \cdot \sigma & = & -(\rho \cdot \sigma) = -(\sigma \cdot \rho) = \sigma \cdot (-\rho), \\ \rho \cdot (-\sigma) & = & -(\rho \cdot \sigma) = -(\sigma \cdot \rho) = (-\sigma) \cdot \rho, \\ (-\rho) \cdot (-\sigma) & = & \rho \cdot \sigma = \sigma \cdot \rho = (-\sigma) \cdot (-\rho). \end{array}$$

Now let α and β be arbitrary left segments of the natural numbers. If either α or β is zero then the products $\alpha \cdot \beta$ and $\beta \cdot \alpha$ both evaluate to the zero segment and are therefore equal to one another. If α and β are both non-zero then there are positive left segments ρ and σ of the rational numbers such that $\alpha = \pm\rho$ and $\beta = \pm\sigma$, and the identity $\alpha \cdot \beta = \beta \cdot \alpha$ reduces to one of the cases checked out above. The result follows. ∎

Proposition A.44 *Let α, β and γ be left segments of the rational numbers. Then*

$$(\alpha \cdot \beta) \cdot \gamma = \alpha \cdot (\beta \cdot \gamma).$$

Proof Let ρ, σ and τ be positive left segments of the rational numbers. Proposition A.33 ensures that $(\rho \cdot \sigma) \cdot \tau = \rho \cdot (\sigma \cdot \tau)$. Then

$$\begin{array}{rcl} ((-\rho) \cdot \sigma) \cdot \tau & = & (-(\rho \cdot \sigma)) \cdot \tau = -((\rho \cdot \sigma)) \cdot \tau) \\ & = & -(\rho \cdot (\sigma \cdot \tau)) = (-\rho) \cdot (\sigma \cdot \tau), \\ (\rho \cdot (-\sigma)) \cdot \tau & = & (-(\rho \cdot \sigma)) \cdot \tau = -((\rho \cdot \sigma) \cdot \tau) \\ & = & -(\rho \cdot (\sigma \cdot \tau)) = \rho \cdot (-(\sigma \cdot \tau)) \end{array}$$

$$
\begin{aligned}
&= \rho \cdot ((-\sigma) \cdot \tau), \\
(\rho \cdot \sigma) \cdot (-\tau) &= -((\rho \cdot \sigma)) \cdot \tau) = -(\rho \cdot (\sigma \cdot \tau)) \\
&= \rho \cdot (-(\sigma \cdot \tau)) = \rho \cdot (\sigma \cdot (-\tau)), \\
(\rho \cdot (-\sigma)) \cdot (-\tau) &= (-(\rho \cdot \sigma)) \cdot (-\tau) = (\rho \cdot \sigma) \cdot \tau \\
&= \rho \cdot (\sigma \cdot \tau) = \rho \cdot ((-\sigma) \cdot (-\tau)), \\
((-\rho) \cdot \sigma) \cdot (-\tau) &= (-(\rho \cdot \sigma)) \cdot (-\tau) = (\rho \cdot \sigma) \cdot \tau \\
&= \rho \cdot (\sigma \cdot \tau) = (-\rho) \cdot (-(\sigma \cdot \tau)) \\
&= (-\rho) \cdot (\sigma \cdot (-\tau)), \\
((-\rho) \cdot (-\sigma)) \cdot \tau &= (\rho \cdot \sigma) \cdot \tau = \rho \cdot (\sigma \cdot \tau) \\
&= (-\rho) \cdot (-(\sigma \cdot \tau)) = (-\rho) \cdot ((-\sigma) \cdot \tau), \\
((-\rho) \cdot (-\sigma)) \cdot (-\tau) &= (\rho \cdot \sigma) \cdot (-\tau) = -((\rho \cdot \sigma) \cdot \tau) \\
&= -(\rho \cdot (\sigma \cdot \tau)) = (-\rho) \cdot (\sigma \cdot \tau) \\
&= (-\rho) \cdot ((-\sigma) \cdot (-\tau)).
\end{aligned}
$$

Now let α, β and γ be arbitrary left segments of the natural numbers. If any of α, β or γ is zero then the products $(\alpha \cdot \beta) \cdot \gamma$ and $\alpha \cdot (\beta \cdot \gamma)$ both evaluate to the zero segment and are therefore equal to one another. If α, β and γ are all non-zero then there are positive left segments ρ, σ and τ of the rational numbers such that $\alpha = \pm\rho$, $\beta = \pm\sigma$ and $\gamma = \pm\tau$, and the identity $(\alpha \cdot \beta) \cdot \gamma = \alpha \cdot (\beta \cdot \gamma)$ reduces to one of the cases checked out above. The result follows. ∎

Proposition A.45 *An operation of multiplication may be defined on the set of all left segments of the rational numbers characterized by the following three conditions:—*

- $\alpha \cdot \beta = \lambda_0$ *if* $\alpha = \lambda_0$ *or* $\beta = \lambda_0$, *where* $\lambda_0 = \{p \in \mathbb{Q} : p < 0\}$*;*
- $(-\alpha) \cdot \beta = \alpha \cdot (-\beta) = -(\alpha \cdot \beta)$ *for all left segments* α *and* β *of the rational numbers;*
- *a rational number* p *belongs to a product* $\alpha \cdot \beta$ *of positive left segments* α *and* β *if and only if there exist positive rational numbers* s *and* t *such that* $s \in \alpha$, $t \in \beta$ *and* $p \leq st$.

This operation of multiplication defined on the set of non-zero left segments of the rational numbers satisfies the following properties:—

- (commutativity of multiplication) $\alpha \cdot \beta = \beta \cdot \alpha$ *for all left segments* α *and* β*;*

- (associativity of multiplication) $(\alpha\cdot\beta)\cdot\gamma = \alpha\cdot(\beta\cdot\gamma)$ *for all left segments* α, β *and* γ*;*
- (multiplicative identity element) $\alpha \cdot \lambda_1 = \lambda_1 \cdot \alpha$ *for all left segments* α*, where*
$$\lambda_1 = \{p \in \mathbb{Q} : p < 1\}.$$
- (existence of reciprocals) *given any non-zero left segment* α *of the rational numbers, there exists a non-zero left segment* α^{-1}*, the reciprocal of* α*, characterized by the property that* $\alpha \cdot \alpha^{-1} = \alpha^{-1} \cdot \alpha = \lambda_1$.

The set consisting of all non-zero left segments of the rational numbers is then a commutative group with respect to the operation of multiplication defined on left segments.

Proof The operation of multiplication defined on left segments is commutative (Proposition A.43) and associative (Proposition A.44). It follows from (Corollary A.35) and (Corollary A.40) that $\lambda_1 \cdot \alpha = \alpha \cdot \lambda_1 = \alpha$ and $\alpha^{-1}\cdot\alpha = \alpha\cdot\alpha^{-1} = \lambda_1$ for all positive left segments α of the rational numbers. The identity $\lambda_1 \cdot \alpha = \alpha \cdot \lambda_1 = \alpha$ also holds when α is equal to the zero segment $\{q \in \mathbb{Q} : q \leq 0\}$, because the product of any left segment with the zero segment is by definition equal to the zero segment. Suppose that α is a negative left segment of the rational numbers. Let $\rho = -\alpha$. Then ρ is a positive segment of the rational numbers, and therefore that $\lambda_1\cdot\rho = \rho\cdot\lambda_1 = \rho$ and $\rho^{-1} \cdot \rho = \rho \cdot \rho^{-1} = \lambda_1$. It follows that

$$\lambda_1 \cdot \alpha = \lambda_1 \cdot (-\rho) = -(\lambda_1 \cdot \rho) = -\rho = \alpha.$$

Similarly $\alpha \cdot \lambda_1\alpha$. Also

$$(-\rho^{-1}) \cdot \alpha = (-\rho^{-1}) \cdot (-\rho) = \rho^{-1} \cdot \rho = \lambda_1,$$

and similarly $\alpha \cdot (-\rho^{-1}) = \lambda_1$. Thus the identity $\alpha^{-1} \cdot \alpha = \alpha \cdot \alpha^{-1} = \lambda_1$ holds with $\alpha^{-1} = -\rho^{-1}$. This completes the proof. ∎

Proposition A.46 *Let* α*,* β *and* γ *be left segments of the rational numbers. Then* $(\alpha + \beta) \cdot \gamma = \alpha \cdot \gamma + \beta \cdot \gamma$.

Proof This result has already been established in the case where α, β and γ are all positive left segments of the rational numbers (see Proposition A.46). The result in other cases will be established below by considering separately the various cases that arise.

Let α, β and γ be left segments for which α and γ are positive, β is negative and $\alpha+\beta$ is zero. Then $\beta = -\alpha$, and the definitions of multiplication of left segments in which some factors are zero or negative ensures that

$$\begin{aligned}(\alpha + \beta) \cdot \gamma &= \lambda_0 \cdot \gamma = \lambda_0 \\ &= (\alpha \cdot \gamma) + (-(\alpha \cdot \gamma)) = (\alpha \cdot \gamma) + (-\alpha) \cdot \gamma \\ &= \alpha \cdot \gamma + \beta \cdot \gamma\end{aligned}$$

in this case.

Next let α, β and γ be left segments for which α and γ are positive, β is negative and $\alpha + \beta$ is positive. Let $\sigma = -\beta$. Then α, σ, γ and $\alpha - \sigma$ are all positive left segments. It then follows from Proposition A.42 that $(\alpha - \sigma) \cdot \gamma = \alpha \cdot \gamma - \sigma \cdot \gamma$. But $\alpha - \sigma = \alpha + \beta$ and $\sigma \cdot \gamma = -\beta \cdot \gamma$. Therefore $(\alpha + \beta) \cdot \gamma = \alpha \cdot \gamma + \beta \cdot \gamma$ in this case.

Next let α, β and γ be left segments for which α and γ are positive, β is negative and $\alpha + \beta$ is negative. Let $\sigma = -\beta$. Then α, σ, γ and $\sigma - \alpha$ are all positive left segments. It then follows from Proposition A.42 that $(\sigma - \alpha) \cdot \gamma = \sigma \cdot \gamma - \alpha \cdot \gamma$. But $\sigma - \alpha = -(\alpha + \beta)$ and $\sigma \cdot \gamma = -\beta \cdot \gamma$. Therefore $(\alpha + \beta) \cdot \gamma = \alpha \cdot \gamma + \beta \cdot \gamma$ in this case.

From the results now verified we see that $(\alpha + \beta) \cdot \gamma = \alpha \cdot \gamma + \beta \cdot \gamma$ in all cases (determined by the sign of $\alpha + \beta$) where α and γ are both positive and β is negative. Similarly this identity holds in all cases where where β and γ are both positive and α is negative.

Next let α, β and γ be left segments for which α and β are both negative and γ is positive. Let $\rho = -\alpha$ and $\sigma = -\beta$. Then ρ, σ and γ are all positive left segments. It then follows from Proposition A.42 that

$$(\rho + \sigma) \cdot \gamma = \rho \cdot \gamma + \sigma \cdot \gamma.$$

But $\rho + \sigma = -(\alpha + \beta)$, and therefore

$$(\rho + \sigma) \cdot \gamma = -(\alpha + \beta) \cdot \gamma.$$

It follows that

$$(\alpha + \beta) \cdot \gamma = -((\rho + \sigma) \cdot \gamma) = (-(\rho \cdot \gamma)) + (-(\sigma \cdot \gamma)) = \alpha \cdot \gamma + \beta \cdot \gamma$$

in this case.

Next let α, β and γ be left segments for which α is the zero segment. Then

$$(\alpha + \beta) \cdot \gamma = \beta \cdot \gamma = \alpha \cdot \gamma + \beta \cdot \gamma.$$

Next let α, β and γ be left segments for which β is the zero segment. Then

$$(\alpha + \beta) \cdot \gamma = \alpha \cdot \gamma = \alpha \cdot \gamma + \beta \cdot \gamma.$$

The cases so far considered establish that $(\alpha + \beta) \cdot \gamma = \alpha \cdot \gamma + \beta \cdot \gamma$ in all cases where α and β are left segments (positive, negative or zero) of the rational numbers and γ is a positive left segment of the rational numbers.

Next let α, β and γ be left segments for which the left segment γ is negative. Let $\tau = -\gamma$. Then τ is a positive left segment, and therefore

$$\begin{aligned}(\alpha + \beta) \cdot \gamma &= -((\alpha + \beta) \cdot \tau) = (-\alpha \cdot \tau) + (\beta \cdot \tau) \\ &= \alpha \cdot \gamma + \beta \cdot \gamma\end{aligned}$$

in this case.

Finally let α, β and γ be left segments for which γ is the zero segment. Then

$$(\alpha + \beta) \cdot \gamma = \lambda_0 = \alpha \cdot \gamma + \beta \cdot \gamma.$$

We are finally in a position to conclude that $(\alpha + \beta) \cdot \gamma = \alpha \cdot \gamma + \beta \cdot \gamma$ for all left segments α, β and γ of the rational numbers, as required. ∎

Theorem A.47 *The set of all left segments of the rational numbers, with the operations of addition and multiplication of left segments defined as described above, is an ordered field.*

Proof It follows from Proposition A.20, Proposition A.45, Proposition A.46, Proposition A.4, Proposition A.5, Proposition A.25 and the definition of multiplication of positive left segments that all axioms required for an ordered field are satisfied by the set of all left segments of rational numbers, with the operations of addition and multiplication (and associated operations of subtraction and division) defined in the manner described above. ∎

A.5 Upper Bounds on Sets of Left Segments

Let S be a set whose elements are left segments of the rational numbers. The set S is said to be *bounded above* if there exists a left segment θ of the rational numbers with the property that $\alpha \leq \theta$ for all $\alpha \in S$. A left segment θ with this property is said to be an *upper bound* for the set S.

Lemma A.48 *Let S be a set whose elements are left segments of the rational numbers. Then S is bounded above if and only if there exists some rational number r with the property that $r \leq q$ for all rational numbers q that belong to left segments in the set S.*

Proof Suppose that the set S is bounded above. Then there exists a left segment θ of the rational numbers with the property that $\alpha \leq \theta$ for all $\alpha \in S$. The definition of the ordering of left segments then ensures that $\alpha \subset \theta$ for all $\alpha \in S$. Property (ii) in the definition of left segments then ensures the existence of a rational number r which does not belong to θ. It then follows from Lemma A.1 that $q \leq r$ for all $q \in \theta$. Then $r \geq q$ for all rational numbers q that belong to left segments in the set S.

Conversely if S is a set of left segments, and if r is a rational number with the property that $r \leq q$ for all rational numbers q that belong to left segments in the set S. Then $\alpha \leq \lambda_r$ for all $\alpha \in S$, where $\lambda_r = \{q \in \mathbb{Q} : q < r\}$, and therefore the set S is bounded above by λ_r. The result follows. ∎

Let S be a subset of the set of left segments of the rational numbers. Suppose that S is non-empty and bounded above. An upper bound for S is said to be the *least upper bound* for S if it is less than or equal to every other upper bound for the set S.

Proposition A.49 *Let S be a subset of the set of left segments of the rational numbers which is non-empty and bounded above. Then the union of the left segments belonging to the set S is a left segment of the rational numbers that is the least upper bound for the set S.*

Proof Let σ be the union of the left segments belonging to the set S. Then σ is non-empty, because S is non-empty and the left segments that belong to S are also non-empty. Thus σ satisfies property (i) in the definition of left segments. Also there exists some rational number r that satisfies $r \geq q$ for all rational numbers q that belong to left segments in the set S. Then $r \geq q$ for all $q \in \sigma$. It follows that σ is not the entire set of rational numbers, and thus σ satisfies property (ii) in the definition of left segments.

Let p and q be rational numbers with $p < q$ and $q \in \sigma$. Then $q \in \alpha$ for some $\alpha \in S$. But then $p \in \alpha$, and therefore $p \in \sigma$. Thus σ satisfies property (ii) in the definition of left segments.

If q is a rational number, and if $q \in \sigma$, then $q \in \alpha$ for some $\alpha \in S$. Then there exists a rational number r such that $r > q$ and $r \in \alpha$. But then $r \in \sigma$. Thus σ satisfies property (iv) in the definition of left segments. We have now shown that σ is a left segment of the rational numbers.

The left segment σ is an upper bound for the set S, because $\alpha \subset \sigma$ for all $\alpha \in S$. Let θ be a left segment of the rational numbers that is an upper bound for the set S. Then $\alpha \subset \theta$ for all $\alpha \in S$, and therefore $\sigma \subset \theta$, because σ is the union of all the left segments belonging to S. It follows that the left segment σ is the least upper bound of the set S, as required. ∎

Corollary A.50 *The set of all left segments of the rational numbers, with the operations of addition and multiplication of left segments defined as described above, is a Dedekind-complete ordered field.*

Proof This result follows immediately on combining the results established in Theorem A.47 and Proposition A.49. ∎