Notes on Real Analysis
Basic Properties of the Riemann Integral

David R. Wilkins

We recall the basic definitions of the Darboux lower and upper sums and the upper and lower Riemann integrals associated with bounded real-valued functions that are defined over closed bounded intervals of the real line.

Definition A *partition* P of an interval $[a, b]$ is a set $\{u_0, u_1, u_2, \ldots, u_N\}$ of real numbers satisfying $a = u_0 < u_1 < u_2 < \cdots < u_{N-1} < u_N = b$.

Given any bounded real-valued function f on $[a, b]$, the *upper sum* (or *upper Darboux sum*) $U(P, f)$ of f for the partition P of $[a, b]$ is defined so that

$$U(P, f) = \sum_{i=1}^{N} M_i(u_i - u_{i-1}),$$

where $M_i = \sup\{f(x) : u_{i-1} \leq x \leq u_i\}$.

Similarly the *lower sum* (or *lower Darboux sum*) $L(P, f)$ of f for the partition P of $[a, b]$ is defined so that

$$L(P, f) = \sum_{i=1}^{N} m_i(u_i - u_{i-1}),$$

where $m_i = \inf\{f(x) : u_{i-1} \leq x \leq u_i\}$.

Now $L(P, f) \leq U(P, f)$. Moreover $\sum_{i=1}^{N}(u_i - u_{i-1}) = b - a$, and therefore

$$m(b - a) \leq L(P, f) \leq U(P, f) \leq M(b - a),$$

for any real numbers m and M satisfying $m \leq f(x) \leq M$ for all $x \in [a, b]$.

Accordingly, given any bounded real-valued function f defined over a closed bounded interval $[a, b]$ in the real line, the set consisting of all the values of all the upper sums determined by the partitions of that interval is a non-empty set of real numbers that is bounded above and below, as is the set consisting of the values of all the lower sums determined by the partitions of the closed bounded interval over which the function is defined. Accordingly there are well defined real numbers $\mathcal{U}\int_a^b f(x)\,dx$ and $\mathcal{L}\int_a^b f(x)\,dx$, referred to as the *upper Riemann integral* and *lower Riemann integral* respectively of the function f on the interval $[a,b]$, where

$$\begin{aligned} \mathcal{U}\int_a^b f(x)\,dx &= \inf\left\{U(P,f) : P \text{ is a partition of } [a,b]\right\}, \\ \mathcal{L}\int_a^b f(x)\,dx &= \sup\left\{L(P,f) : P \text{ is a partition of } [a,b]\right\}. \end{aligned}$$

A bounded function $f\colon [a,b] \to \mathbb{R}$ on a closed bounded interval $[a,b]$ is said to be *Riemann-integrable* (or *Darboux-integrable*) on $[a,b]$ if

$$\mathcal{U}\int_a^b f(x)\,dx = \mathcal{L}\int_a^b f(x)\,dx,$$

in which case the *Riemann integral* $\int_a^b f(x)\,dx$ (or *Darboux integral*) of f on $[a,b]$ is defined to be the common value of $\mathcal{U}\int_a^b f(x)\,dx$ and $\mathcal{L}\int_a^b f(x)\,dx$.

Now if f and g are bounded Riemann-integrable functions on the interval $[a,b]$, and if $f(x) \le g(x)$ for all $x \in [a,b]$, then $\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$, since $L(P,f) \le L(P,g)$ and $U(P,f) \le U(P,g)$ for all partitions P of $[a,b]$.

Definition Let P and R be partitions of $[a,b]$, given by $P = \{u_0, u_1, \ldots, u_N\}$ and $R = \{v_0, v_1, \ldots, v_L\}$. We say that the partition R is a *refinement* of P if $P \subset R$, so that, for each u_i in P, there is some v_j in R with $u_i = v_j$.

Lemma A. *Let R be a refinement of some partition P of $[a,b]$. Then*

$$L(R,f) \ge L(P,f) \quad \textit{and} \quad U(R,f) \le U(P,f)$$

for any bounded function $f\colon [a,b] \to \mathbb{R}$.

Proof Let $P = \{u_0, u_1, \ldots, u_N\}$ and $R = \{v_0, v_1, \ldots, v_L\}$, where $a = u_0 < u_1 < \cdots < u_N = b$ and $a = v_0 < v_1 < \cdots < v_L = b$. Now for each integer i between 0 and N there exists some integer $j(i)$ between 0 and L such that $u_i = v_{j(i)}$ for each i, since R is a refinement of P. Moreover $0 = j(0) < j(1) < \cdots < j(N) = L$. For each i, let R_i be the partition of

$[u_{i-1}, u_i]$ given by $R_i = \{v_j : j(i-1) \leq j \leq j(i)\}$. Then $L(R, f) = \sum_{i=1}^{N} L(R_i, f)$ and $U(R, f) = \sum_{i=1}^{N} U(R_i, f)$. Moreover

$$m_i(u_i - u_{i-1}) \leq L(R_i, f) \leq U(R_i, f) \leq M_i(u_i - u_{i-1}),$$

since $m_i \leq f(x) \leq M_i$ for all $x \in [u_{i-1}, u_i]$. On summing these inequalities over i, we deduce that $L(P, f) \leq L(R, f) \leq U(R, f) \leq U(P, f)$, as required. ∎

Given any two partitions P and Q of $[a, b]$ there exists a partition R of $[a, b]$ which is a refinement of both P and Q. For example, we can take $R = P \cup Q$. Such a partition is said to be a *common refinement* of the partitions P and Q.

Lemma B. *Let f be a bounded real-valued function on the interval $[a, b]$. Then*

$$\mathcal{L}\int_a^b f(x)\,dx \leq \mathcal{U}\int_a^b f(x)\,dx.$$

Proof Let P and Q be partitions of $[a, b]$, and let R be a common refinement of P and Q. It follows from Lemma A that $L(P, f) \leq L(R, f) \leq U(R, f) \leq U(Q, f)$. Thus, on taking the supremum of the left hand side of the inequality $L(P, f) \leq U(Q, f)$ as P ranges over all possible partitions of the interval $[a, b]$, we see that $\mathcal{L}\int_a^b f(x)\,dx \leq U(Q, f)$ for all partitions Q of $[a, b]$. But then, taking the infimum of the right hand side of this inequality as Q ranges over all possible partitions of $[a, b]$, we see that $\mathcal{L}\int_a^b f(x)\,dx \leq \mathcal{U}\int_a^b f(x)\,dx$, as required. ∎

Example Let $f(x) = cx + d$, where $c \geq 0$. We shall show that f is Riemann-integrable on $[0, 1]$ and evaluate $\int_0^1 f(x)\,dx$ from first principles.

For each positive integer N, let P_N denote the partition of $[0, 1]$ into N subintervals of equal length. Thus $P_N = \{u_0, u_1, \ldots, u_N\}$, where $u_i = i/N$. Now the function f takes values between $(i-1)c/N + d$ and $ic/N + d$ on the interval $[u_{i-1}, u_i]$, and therefore

$$m_i = \frac{(i-1)c}{N} + d, \qquad M_i = \frac{ic}{N} + d$$

where $m_i = \inf\{f(x) : u_{i-1} \leq x \leq u_i\}$ and $M_i = \sup\{f(x) : u_{i-1} \leq x \leq u_i\}$. Thus

$$\begin{aligned}
L(P_N, f) &= \sum_{i=1}^{N} m_i(u_i - u_{i-1}) = \frac{1}{N}\sum_{i=1}^{N}\left(\frac{ci}{N} + d - \frac{c}{N}\right) \\
&= \frac{c(N+1)}{2N} + d - \frac{c}{N} = \frac{c}{2} + d - \frac{c}{2N}, \\
U(P_N, f) &= \sum_{i=1}^{N} M_i(u_i - u_{i-1}) = \frac{1}{N}\sum_{i=1}^{N}\left(\frac{ci}{N} + d\right) \\
&= \frac{c(N+1)}{2N} + d = \frac{c}{2} + d + \frac{c}{2N}.
\end{aligned}$$

It follows that

$$\lim_{N\to+\infty} L(P_N, f) = \frac{c}{2} + d$$

and

$$\lim_{N\to+\infty} U(P_N, f) = \frac{c}{2} + d$$

Now $L(P_N, f) \leq \mathcal{L}\int_a^b f(x)\,dx \leq \mathcal{U}\int_a^b f(x)\,dx \leq U(P_N, f)$ for all positive integers N. It follows that $\mathcal{L}\int_a^b f(x)\,dx = \frac{1}{2}c + d = \mathcal{U}\int_a^b f(x)\,dx$. Thus f is Riemann-integrable on the interval $[0,1]$, and $\int_0^1 f(x)\,dx = \frac{1}{2}c + d$.

Example Let $f\colon [0,1] \to \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational;} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

Let P be a partition of the interval $[0,1]$ given by $P = \{u_0, u_1, u_2, \ldots, u_N\}$, where $0 = u_0 < u_1 < u_2 < \cdots < u_N = 1$. Then

$$\inf\{f(x) : u_{i-1} \leq x \leq u_i\} = 0, \qquad \sup\{f(x) : u_{i-1} \leq x \leq u_i\} = 1,$$

for $i = 1, 2, \ldots, N$, and thus $L(P, f) = 0$ and $U(P, f) = 1$ for all partitions P of the interval $[0,1]$. It follows that $\mathcal{L}\int_0^1 f(x)\,dx = 0$ and $\mathcal{U}\int_0^1 f(x)\,dx = 1$, and therefore the function f is not Riemann-integrable on the interval $[0,1]$.

0.1 Basic Properties of the Riemann Integral

Lemma C. *Let $f\colon [a,b] \to \mathbb{R}$ be a bounded function on a closed bounded interval $[a,b]$, where a and b are real numbers satisfying $a \leq b$. Then the lower and upper Riemann integrals of f and $-f$ are related by the identities*

$$\begin{aligned}
\mathcal{U}\int_a^b (-f(x))\,dx &= -\mathcal{L}\int_a^b f(x)\,dx, \\
\mathcal{L}\int_a^b (-f(x))\,dx &= -\mathcal{U}\int_a^b f(x)\,dx.
\end{aligned}$$

Proof Let P be a partition of $[a, b]$, and let $P = \{u_0, u_1, u_2, \ldots, u_N\}$, where

$$a = u_0 < u_1 < u_2 < \cdots < u_N = b.$$

Also let

$$\begin{aligned} m_i[f] &= \inf\{f(x) : u_{i-1} \le x \le u_i\}, \\ M_i[f] &= \sup\{f(x) : u_{i-1} \le x \le u_i\}, \\ m_i[-f] &= \inf\{-f(x) : u_{i-1} \le x \le u_i\}, \\ M_i[-f] &= \sup\{-f(x) : u_{i-1} \le x \le u_i\} \end{aligned}$$

for $i = 1, 2, \ldots, N$. Then $m_i[-f] = -M_i[f]$ and $M_i[-f] = -m_i[f]$, and therefore

$$\begin{aligned} L(P, -f) &= \sum_{i=1}^{N} m_i[-f](u_i - u_{i-1}) = -\sum_{i=1}^{N} M_i[f](u_i - u_{i-1}) \\ &= -U(P, f). \end{aligned}$$

Thus $L(P, -f) = -U(P, f)$ for all partitions P of the interval $[a, b]$. Similarly $U(P, -f) = -L(P, f)$ for all partitions P of that interval. It follows from the definition of the upper and lower integrals that

$$\begin{aligned} \mathcal{U}\int_a^b (-f(x))\,dx &= \inf\{U(P, -f) : P \text{ is a partition of } [a, b]\} \\ &= \inf\{-L(P, f) : P \text{ is a partition of } [a, b]\} \\ &= -\sup\{L(P, f) : P \text{ is a partition of } [a, b]\} \\ &= -\mathcal{L}\int_a^b f(x)\,dx \end{aligned}$$

Similarly

$$\begin{aligned} \mathcal{L}\int_a^b (-f(x))\,dx &= \sup\{L(P, -f) : P \text{ is a partition of } [a, b]\} \\ &= \sup\{-U(P, f) : P \text{ is a partition of } [a, b]\} \\ &= -\inf\{U(P, f) : P \text{ is a partition of } [a, b]\} \\ &= -\mathcal{U}\int_a^b f(x)\,dx. \end{aligned}$$

This completes the proof. ∎

Proposition D. *Let $f\colon [a,b] \to \mathbb{R}$ and $g\colon [a,b] \to \mathbb{R}$ be bounded Riemann-integrable functions on a closed bounded interval $[a,b]$, where a and b are real numbers satisfying $a \leq b$. Then the functions $f+g$ and $f-g$ are Riemann-integrable on $[a,b]$, and moreover*

$$\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx,$$

and

$$\int_a^b (f(x)-g(x))\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx.$$

Proof Let some strictly positive real number ε be given. The definition of Riemann-integrability and the Riemann integral ensures that there exist partitions P_1, P_2, P_3 and P_4 of $[a,b]$ for which

$$L(P_1,f) > \int_a^b f(x)\,dx - \tfrac{1}{2}\varepsilon,$$

$$U(P_2,f) < \int_a^b f(x)\,dx + \tfrac{1}{2}\varepsilon,$$

$$L(P_3,g) > \int_a^b g(x)\,dx - \tfrac{1}{2}\varepsilon$$

and

$$U(P_4,g) < \int_a^b g(x)\,dx + \tfrac{1}{2}\varepsilon.$$

Let P be a common refinement of P_1, P_2, P_3 and P_4. Applying Lemma A, we see that

$$L(P,f) \geq L(P_1,f) > \int_a^b f(x)\,dx - \tfrac{1}{2}\varepsilon,$$

$$U(P,f) \leq U(P_2,f) < \int_a^b f(x)\,dx + \tfrac{1}{2}\varepsilon,$$

$$L(P,g) \geq L(P_3,g) > \int_a^b g(x)\,dx - \tfrac{1}{2}\varepsilon$$

and

$$U(P,g) \leq U(P_4,g) < \int_a^b g(x)\,dx + \tfrac{1}{2}\varepsilon.$$

Let $P = \{u_0, u_1, \ldots, u_N\}$, where

$$a = u_0 < u_1 < \cdots < u_N = b,$$

and let

$$\begin{aligned}
M_i[f] &= \sup\{f(x) : u_{i-1} \le x \le u_i\}, \\
m_i[f] &= \inf\{f(x) : u_{i-1} \le x \le u_i\}, \\
M_i[g] &= \sup\{g(x) : u_{i-1} \le x \le u_i\}, \\
m_i[g] &= \inf\{g(x) : u_{i-1} \le x \le u_i\}, \\
M_i[f+g] &= \sup\{f(x) + g(x) : u_{i-1} \le x \le u_i\}, \\
m_i[f+g] &= \inf\{f(x) + g(x) : u_{i-1} \le x \le u_i\}.
\end{aligned}$$

Now the inequalities

$$m_i[f] + m_i[g] \le f(x) + g(x) \le M_i[f] + M_i[g]$$

are satisfied for $i = 1, 2, \ldots, N$ and for all $x \in [u_{i-1}, u_i]$. It follows from the definitions of $M_i[f+g]$ and $m_i[f+g]$ as the least upper bound and greatest lower bound respectively of the values of $f(x)+g(x)$ on the interval $[u_{i-1}, u_i]$ that

$$m_i[f] + m_i[g] \le m_i[f+g] \le M_i[f+g] \le M_i[f] + M_i[g]$$

for $i = 1, 2, \ldots, N$. Multiplying these inequalities by the lengths $u_i - u_{i-1}$ of the subintervals determined by the partition P, and then summing over $i = 1, 2, \ldots, N$, we deduce that

$$L(P, f) + L(P, g) \le L(P, f+g) \le U(P, f+g) \le U(P, f) + U(P, g).$$

Now inequalities satisfied by the Darboux upper and lower sums for the partition P guaranteed by the choice of P (as described above) then ensure that

$$L(P, f) + L(P, g) > \int_a^b f(x)\,dx + \int_a^b g(x)\,dx - \varepsilon$$

and

$$U(P, f) + U(P, g) < \int_a^b f(x)\,dx + \int_a^b g(x)\,dx + \varepsilon.$$

It follows that

$$\begin{aligned}
\int_a^b f(x)\,dx + \int_a^b g(x)\,dx - \varepsilon &< L(P, f+g) \le U(P, f+g) \\
&< \int_a^b f(x)\,dx + \int_a^b g(x)\,dx + \varepsilon.
\end{aligned}$$

But

$$\begin{aligned} L(P, f+g) &\leq \mathcal{L}\int_a^b (f(x)+g(x))\,dx \\ &\leq \mathcal{U}\int_a^b (f(x)+g(x))\,dx \leq U(P, f+g). \end{aligned}$$

It follows therefore that

$$\begin{aligned} \int_a^b f(x)\,dx + \int_a^b g(x)\,dx - \varepsilon &< \mathcal{L}\int_a^b (f(x)+g(x))\,dx \\ &\leq \mathcal{U}\int_a^b (f(x)+g(x))\,dx \\ &< \int_a^b f(x)\,dx + \int_a^b g(x)\,dx + \varepsilon. \end{aligned}$$

These latter inequalities must hold for all positive real numbers ε, no matter how small. It follows that

$$\begin{aligned} \int_a^b f(x)\,dx + \int_a^b g(x)\,dx &\leq \mathcal{L}\int_a^b (f(x)+g(x))\,dx \\ &\leq \mathcal{U}\int_a^b (f(x)+g(x))\,dx \\ &\leq \int_a^b f(x)\,dx + \int_a^b g(x)\,dx. \end{aligned}$$

The extreme left hand and extreme right hand sides of the above chain of inequalities are equal. Therefore

$$\begin{aligned} \mathcal{L}\int_a^b (f(x)+g(x))\,dx &= \mathcal{U}\int_a^b (f(x)+g(x))\,dx \\ &= \int_a^b f(x)\,dx + \int_a^b g(x)\,dx. \end{aligned}$$

We conclude therefore that the function $f+g$ is Riemann-integrable and that the value of the Riemann integral of this function is the sum of the integrals of the functions f and g on the interval $[a,b]$.

On replacing g by $-g$, we may deduce the corresponding result for the function $f-g$, thereby completing the proof. ∎

Proposition E. *Let $f\colon [a,b] \to \mathbb{R}$ be a bounded function on a closed bounded interval $[a,b]$, where a and b are real numbers satisfying $a \leq b$. Then the* *function f is Riemann-integrable on $[a,b]$ if and only if, given any positive real number ε, there exists a partition P of $[a,b]$ with the property that*

$$U(P,f) - L(P,f) < \varepsilon.$$

Proof First suppose that $f\colon [a,b] \to \mathbb{R}$ is Riemann-integrable on $[a,b]$. Let some positive real number ε be given. Then

$$\int_a^b f(x)\,dx$$

is equal to the common value of the lower and upper integrals of the function f on $[a,b]$, and therefore there exist partitions Q and R of $[a,b]$ for which

$$L(Q,f) > \int_a^b f(x)\,dx - \tfrac{1}{2}\varepsilon$$

and

$$U(R,f) < \int_a^b f(x)\,dx + \tfrac{1}{2}\varepsilon.$$

Let P be a common refinement of the partitions Q and R. Now

$$L(Q,f) \leq L(P,f) \leq U(P,f) \leq U(R,f).$$

(see Lemma A). It follows that

$$U(P,f) - L(P,f) \leq U(R,f) - L(Q,f) < \varepsilon.$$

Now suppose that $f\colon [a,b] \to \mathbb{R}$ is a bounded function on $[a,b]$ with the property that, given any positive real number ε, there exists a partition P of $[a,b]$ for which $U(P,f) - L(P,f) < \varepsilon$. Let $\varepsilon > 0$ be given. Then there exists a partition P of $[a,b]$ for which $U(P,f) - L(P,f) < \varepsilon$. Now it follows from the definitions of the upper and lower integrals that

$$L(P,f) \leq \mathcal{L}\int_a^b f(x)\,dx \leq \mathcal{U}\int_a^b f(x)\,dx \leq U(P,f),$$

and therefore

$$\mathcal{U}\int_a^b f(x)\,dx - \mathcal{L}\int_a^b f(x)\,dx < U(P,f) - L(P,f) < \varepsilon.$$

Thus the difference between the values of the upper and lower integrals of f on $[a,b]$ must be less than every strictly positive real number ε, and therefore

$$\mathcal{U}\int_a^b f(x)\,dx = \mathcal{L}\int_a^b f(x)\,dx.$$

This completes the proof. ∎

Lemma F. *Let $f_1, f_2, \ldots, f_s$ and h be bounded real-valued functions on a closed bounded interval $[a, b]$, where a and b are real numbers satisfying $a < b$. Suppose that there exists a positive constant K with the property that*

$$|h(v) - h(w)| \leq K \sum_{j=1}^{s} |f_j(v) - f_j(w)|$$

for all $v, w \in [a, b]$. Then the upper and lower Darboux sums of these real-valued functions satisfy the inequality

$$U(P, h) - L(P, h) \leq K \sum_{j=1}^{s} (U(P, f_j) - L(P, f_j))$$

for all partitions P of the interval $[a, b]$.

Proof Let P be a partition of $[a, b]$, and let $P = \{u_0, u_1, \ldots, u_N\}$, where

$$a = u_0 < u_1 < \cdots < u_N = b,$$

and let

$$\begin{aligned} M_i[f_j] &= \sup\{f_j(x) : u_{i-1} \leq x \leq u_i\}, \\ m_i[f_j] &= \inf\{f_j(x) : u_{i-1} \leq x \leq u_i\} \end{aligned}$$

for $j = 1, 2, \ldots, s$ and $i = 1, 2, \ldots, N$, and

$$\begin{aligned} M_i[h] &= \sup\{h(x) : u_{i-1} \leq x \leq u_i\}, \\ m_i[h] &= \inf\{h(x) : u_{i-1} \leq x \leq u_i\} \end{aligned}$$

for $i = 1, 2, \ldots, N$.

Let i be an integer between 1 and N. The definitions of $M_i[h]$ and $m_i[h]$ ensure that, given any positive real number δ, there exist $v_i, w_i \in [u_{i-1}, u_i]$ such that $h(v_i) > M_i[h] - \delta$ and $h(w_i) < m_i[h] + \delta$. But then

$$\begin{aligned} M_i[h] - m_i[h] - 2\delta &< h(v_i) - h(w_i) \leq K \sum_{j=1}^{s} |f_j(v_i) - f_j(w_i)| \\ &\leq K \sum_{j=1}^{s} (M_i[f_j] - m_i[f_j]). \end{aligned}$$

The inequality

$$M_i[h] - m_i[h] - 2\delta < K \sum_{j=1}^{s} (M_i[f_j] - m_i[f_j])$$

therefore holds for all positive values of the real number δ, no matter how small, and therefore

$$M_i[h] - m_i[h] \le K \sum_{j=1}^{s} (M_i[f_j] - m_i[f_j]).$$

Multiplying both sides of this inequality by the length $u_i - u_{i-1}$ of the ith subinterval of $[a, b]$ determined by the partition P, and summing for $i = 1, 2, \ldots, N$, we find that

$$\begin{aligned} U(P, h) - L(P, h) &= \sum_{i=1}^{N} (M_i[h] - m_i[h])(u_i - u_{i-1}) \\ &\le K \sum_{j=1}^{s} \sum_{i=1}^{N} (M_i[f_j] - m_i[f_j])(u_i - u_{i-1}) \\ &\le K \sum_{j=1}^{s} (U(P, f_j) - L(P, f_j)). \end{aligned}$$

We conclude therefore that

$$U(P, h) - L(P, h) \le K \sum_{j=1}^{s} (U(P, f_j) - L(P, f_j))$$

for all partitions P of the interval $[a, b]$, as required. ∎

Proposition G. *Let $f_1, f_2, \ldots, f_s$ be bounded Riemann-integrable real-valued functions on a closed bounded interval $[a, b]$, where a and b are real numbers satisfying $a < b$, and let h be a bounded real-valued function on $[a, b]$. Suppose that there exists a positive constant K with the property that*

$$|h(v) - h(w)| \le K \sum_{j=1}^{s} |f_j(v) - f_j(w)|$$

for all $u, v \in [a, b]$. Then the function h is Riemann-integrable on $[a, b]$.

Proof Given any positive real number ε, there exist partitions $P_1, P_2, \ldots, P_s$ of $[a, b]$ with the property that

$$U(P_j, f_j) - L(P_j, f_j) < \frac{\varepsilon}{sK}$$

for $j = 1, 2, \ldots, s$ (see Proposition E). Let P be a common refinement of the partitions $P_1, P_2, \ldots, P_s$. Then

$$U(P, f_j) - L(P, f_j) \leq U(P_j, f_j) - L(P_j, f_j) < \frac{\varepsilon}{sK}$$

for $j = 1, 2, \ldots, s$ (see Lemma A). It then follows from Lemma F that

$$U(P, h) - L(P, h) \leq K \sum_{j=1}^{s} (U(P, f_j) - L(P, f_j)) < \varepsilon.$$

On applying Proposition E, we therefore conclude that the function h is Riemann-integrable on $[a, b]$, as required. ∎

Proposition H. *Let $f\colon [a, b] \to \mathbb{R}$ and $g\colon [a, b] \to \mathbb{R}$ be bounded Riemann-integrable functions on a closed bounded interval $[a, b]$, where a and b are real numbers satisfying $a \leq b$. Then the function $f \cdot g$ is Riemann-integrable on $[a, b]$, where $(f \cdot g)(x) = f(x)g(x)$ for all $x \in [a, b]$.*

Proof The functions f and g are bounded on $[a, b]$, and therefore there exists some positive real number K with the property that $|f(x)| \leq K$ and $|g(x)| \leq K$ for all $x \in [a, b]$. But then

$$\begin{aligned} |f(v)g(v) - f(w)g(w)| &= |f(v)(g(v) - g(w)) + (f(v) - f(w))g(w)| \\ &\leq |f(v)(g(v) - g(w))| + |(f(v) - f(w))g(w)| \\ &\leq K\left(|g(v) - g(w)| + |f(v) - f(w)|\right) \end{aligned}$$

for all $v, w \in [a, b]$. The result therefore follows directly on applying Proposition G. ∎

Proposition I. *Let $f\colon [a, b] \to \mathbb{R}$ be a bounded Riemann-integrable function on a closed interval $[a, b]$, where a and b are real numbers satisfying $a \leq b$, and let $|f|\colon [a, b] \to \mathbb{R}$ be the function defined such that $|f|(x) = |f(x)|$ for all $x \in [a, b]$. Then the function $|f|$ is Riemann-integrable on $[a, b]$, and*

$$\left| \int_a^b f(x)\, dx \right| \leq \int_a^b |f(x)|\, dx.$$

Proof Let some positive real number ε be given. It follows from Proposition E that there exists a partition P of $[a, b]$ such that

$$U(P, f) - L(P, f) < \varepsilon.$$

$$\Big||f(v)| - |f(w)|\Big| \leq |f(v) - f(w)|$$

for all $v, w \in [a, b]$. Applying Lemma F, we conclude that

$$U(P, |f|) - L(P, |f|) \leq U(P, f) - L(P, f) < \varepsilon.$$

Proposition E then ensures that the function $|f|$ is Riemann-integrable on $[a, b]$,

Now $-|f(x)| \leq f(x) \leq |f(x)|$ for all $x \in [a, b]$. It follows that

$$-\int_a^b |f(x)|\,dx \leq \int_a^b f(x)\,dx \leq \int_a^b |f(x)|\,dx.$$

It follows that

$$\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx,$$

as required. ∎

Proposition J. *Let f be a bounded real-valued function on the interval $[a, c]$. Suppose that f is Riemann-integrable on the intervals $[a, b]$ and $[b, c]$, where $a < b < c$. Then f is Riemann-integrable on $[a, c]$, and*

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx.$$

Proof Let some positive real number ε be given. The function f is Riemann-integrable on the interval $[a, b]$ and therefore there exists a partition Q of $[a, b]$ such that the lower Darboux sum $L(Q, f)$ of f on $[a, b]$ with respect to the partition Q of $[a, b]$ satisfies

$$L(Q, f) > \int_a^b f(x)\,dx - \tfrac{1}{2}\varepsilon.$$

Similarly there exists a partition R of $[b, c]$ of $[a, b]$ such that the lower Darboux sum $L(Q, f)$ of f on $[b, c]$ with respect to the partition R of $[b, c]$ satisfies

$$L(R, f) > \int_b^c f(x)\,dx - \tfrac{1}{2}\varepsilon.$$

Now the partitions Q and R combine to give a partition P of the interval $[a, c]$, where $P = Q \cup R$. Indeed $Q = \{v_0, v_1, \ldots, v_L\}$, where $v_0, v_1, \ldots, v_L$ are real numbers satisfying

$$a = v_0 < v_1 < v_2 < \cdots v_{L-1} < v_L = b,$$

and $R = \{w_0, w_1, \ldots, w_N\}$, where $w_0, w_1, \ldots, w_N$ are real numbers satisfying

$$b = w_0 < w_1 < w_2 < \cdots w_{N-1} < w_N = c.$$

Then

$$P = \{a, v_1, v_2, \ldots, v_{L-1}, b, w_1, w_2, \ldots, w_{N-1}, c\}.$$

It follows directly from the definition of Darboux lower sums that

$$L(P, f) = L(Q, f) + L(R, f).$$

The choice of the partitions Q and R then ensures that

$$L(P, f) > \int_a^b f(x)\,dx + \int_b^c f(x)\,dx - \varepsilon.$$

The lower Riemann integral $\mathcal{L}\displaystyle\int_a^c f(x)\,dx$ is by definition the least upper bound of the lower Darboux sums of f on the interval $[a, c]$. It follows that

$$\mathcal{L}\int_a^c f(x)\,dx > \int_a^b f(x)\,dx + \int_b^c f(x)\,dx - \varepsilon.$$

Moreover this inequality holds for all values of the positive real number ε. It follows that

$$\mathcal{L}\int_a^c f(x)\,dx \geq \int_a^b f(x)\,dx + \int_b^c f(x)\,dx.$$

Applying this result with the function f replaced by $-f$ yields the inequality

$$\mathcal{L}\int_a^c (-f(x))\,dx \geq -\int_a^b f(x)\,dx - \int_b^c f(x)\,dx.$$

But

$$\mathcal{L}\int_a^c (-f(x))\,dx = -\mathcal{U}\int_a^c f(x)\,dx$$

(see Lemma C). It follows that

$$\mathcal{U}\int_a^c f(x)\,dx \leq \int_a^b f(x)\,dx + \int_b^c f(x)\,dx \leq \mathcal{L}\int_a^c f(x)\,dx.$$

But

$$\mathcal{L}\int_a^c f(x)\,dx \leq \mathcal{U}\int_a^c f(x)\,dx.$$

It follows that

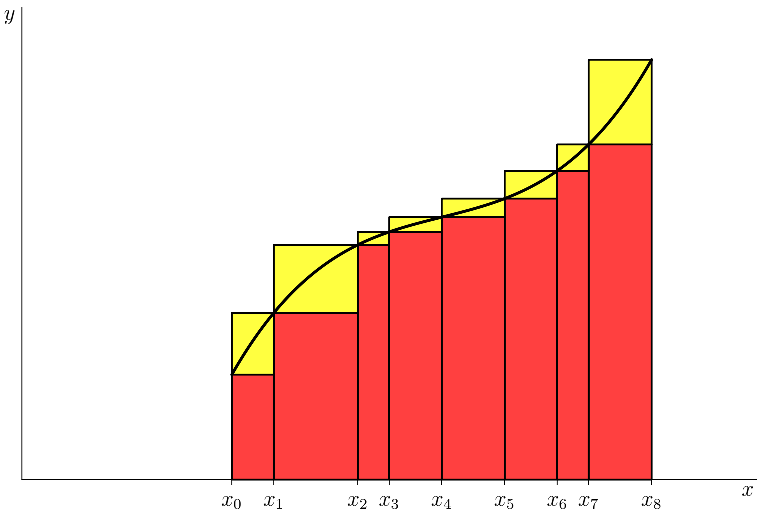
$$\mathcal{L}\int_a^c f(x)\,dx = \mathcal{U}\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx.$$

The result follows. ∎

0.2 Integrability of Monotonic Functions

Let a and b be real numbers satisfying $a < b$. A real-valued function $f: [a, b] \to \mathbb{R}$ defined on the closed bounded interval $[a, b]$ is said to be *non-decreasing* if $f(v) \leq f(w)$ for all real numbers v and w satisfying $a \leq v \leq w \leq b$. Similarly $f: [a, b] \to \mathbb{R}$ is said to be *non-increasing* if $f(v) \geq f(w)$ for all real numbers v and w satisfying $a \leq v \leq w \leq b$. The function $f: [a, b] \to \mathbb{R}$ is said to be *monotonic* on $[a, b]$ if either it is non-decreasing on $[a, b]$ or else it is non-increasing on $[a, b]$.

Proposition K. *Let a and b be real numbers satisfying $a < b$. Then every monotonic function on the interval $[a, b]$ is Riemann-integrable on $[a, b]$.*



Proof Let $f: [a, b] \to \mathbb{R}$ be a non-decreasing function on the closed bounded interval $[a, b]$. Then $f(a) \leq f(x) \leq f(b)$ for all $x \in [a, b]$, and therefore the function f is bounded on $[a, b]$. Let some positive real number ε be given. Let δ be some strictly positive real number for which $(f(b) - f(a))\delta < \varepsilon$, and let P be a partition of $[a, b]$ of the form $P = \{u_0, u_1, u_2, \ldots, u_N\}$, where

$$a = u_0 < u_1 < u_2 < \cdots < u_{N-1} < u_N = b$$

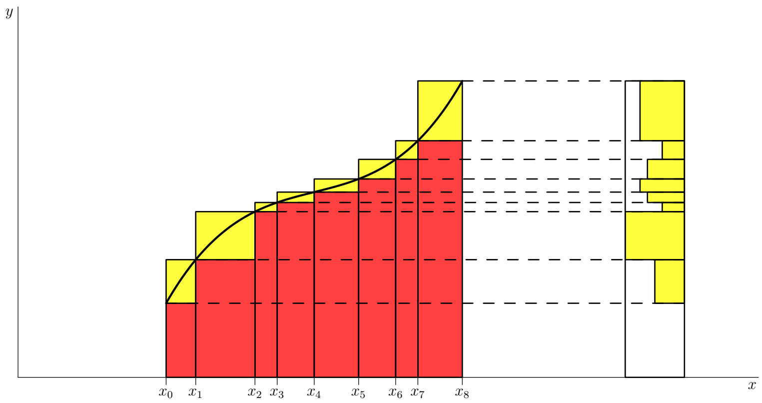
and $u_i - u_{i-1} < \delta$ for $i = 1, 2, \ldots, N$. The maximum and minimum values of $f(x)$ on the interval $[u_{i-1}, u_i]$ are attained at u_i and u_{i-1} respectively, and therefore the upper sum $U(P, f)$ and $L(P, f)$ of f for the partition P satisfy

$$U(P, f) = \sum_{i=1}^{N} f(u_i)(u_i - u_{i-1})$$

and

$$L(P,f) = \sum_{i=1}^{N} f(u_{i-1})(u_i - u_{i-1}).$$

Now $f(u_i) - f(u_{i-1}) \geq 0$ for $i = 1, 2, \ldots, N$. It follows that



$$\begin{aligned} & U(P,f) - L(P,f) \\ = & \sum_{i=1}^{N} (f(u_i) - f(u_{i-1}))(u_i - u_{i-1}) \\ < & \delta \sum_{i=1}^{N} (f(u_i) - f(u_{i-1})) = \delta(f(b) - f(a)) < \varepsilon. \end{aligned}$$

We have thus shown that

$$\mathcal{U}\int_a^b f(x)\,dx - \mathcal{L}\int_a^b f(x)\,dx < \varepsilon$$

for all strictly positive numbers ε. But

$$\mathcal{U}\int_a^b f(x)\,dx \geq \mathcal{L}\int_a^b f(x)\,dx.$$

It follows that

$$\mathcal{U}\int_a^b f(x)\,dx = \mathcal{L}\int_a^b f(x)\,dx,$$

and thus the function f is Riemann-integrable on $[a,b]$.

Now let $f\colon [a,b] \to \mathbb{R}$ be a non-increasing function on $[a,b]$. Then $-f$ is a non-decreasing function on $[a,b]$ and it follows from what we have just shown that $-f$ is Riemann-integrable on $[a,b]$. It follows that the function f itself must be Riemann-integrable on $[a,b]$, as required. ∎

Corollary L. *Let* $f\colon [a,b] \to \mathbb{R}$ *be a real-valued function on the interval* $[a,b]$*, where* a *and* b *are real numbers satisfying* $a < b$*. Suppose that there exist real numbers* $u_0, u_1, \ldots, u_N$*, where*

$$a = u_0 < u_1 < u_2 < \cdots < u_{N-1} < u_N = b,$$

such that the function f *restricted to the interval* $[u_{i-1}, u_i]$ *is monotonic on* $[u_{i-1}, u_i]$ *for* $i = 1, 2, \ldots, N$*. Then* f *is Riemann-integrable on* $[a,b]$*.*

Proof The result follows immediately on applying the results of Proposition J and Proposition K. ∎

Remark The result and proof-strategy of Proposition K are to be found in their essentials in Isaac Newton, *Philosophiae naturalis principia mathematica* (1686), Book 1, Section 1, Lemmas 2 and 3.