MAU23203—Analysis in Several Variables
School of Mathematics, Trinity College
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Section 11: Multiple Integrals

Trinity College Dublin

11. Multiple Integrals

11.1. Darboux Sums and the Riemann Integral

We now set out the basic definitions and state some basic results concerning the theory of integration of functions of a real variable that was developed by Jean-Gaston Darboux (1842–1917). The integral defined using lower and upper sums in the manner described below is sometimes referred to as the *Darboux integral* of a function on a given interval. However the class of functions that are integrable according to the definitions introduced by Darboux is the class of *Riemann-integrable* functions. Thus the approach using Darboux sums provides a convenient approach to define and establish the basic properties of the *Riemann integral*.

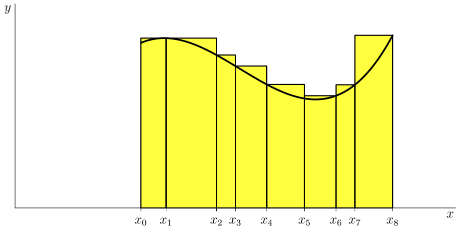
Definition

A *partition* P of an interval $[a, b]$ is a set $\{u_0, u_1, u_2, \ldots, u_N\}$ of real numbers satisfying $a = u_0 < u_1 < u_2 < \cdots < u_{N-1} < u_N = b$.

Given any bounded real-valued function f on $[a, b]$, the *upper sum* (or *upper Darboux sum*) $U(P, f)$ of f for the partition P of $[a, b]$ is defined so that

$$U(P, f) = \sum_{i=1}^{N} M_i(u_i - u_{i-1}),$$

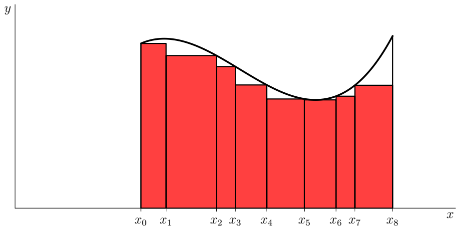
where $M_i = \sup\{f(x) : u_{i-1} \leq x \leq u_i\}$.



Similarly the *lower sum* (or *lower Darboux sum*) $L(P, f)$ of f for the partition P of $[a, b]$ is defined so that

$$L(P, f) = \sum_{i=1}^{N} m_i(u_i - u_{i-1}),$$

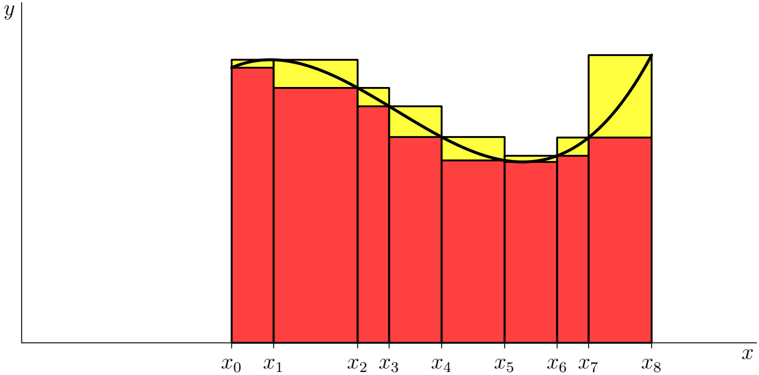
where $m_i = \inf\{f(x) : u_{i-1} \leq x \leq u_i\}$.



Clearly $L(P, f) \leq U(P, f)$. Moreover $\sum_{i=1}^{N}(u_i - u_{i-1}) = b - a$, and therefore

$$m(b - a) \leq L(P, f) \leq U(P, f) \leq M(b - a),$$

for any real numbers m and M satisfying $m \leq f(x) \leq M$ for all $x \in [a, b]$.



y
x
x_0
x_1
x_2
x_3
x_4
x_5
x_6
x_7
x_8

Definition

Let f be a bounded real-valued function on the interval $[a, b]$, where $a < b$. The *upper Riemann integral* $\mathcal{U}\int_a^b f(x)\,dx$ (or *upper Darboux integral*) and the *lower Riemann integral* $\mathcal{L}\int_a^b f(x)\,dx$ (or *lower Darboux integral*) of the function f on $[a, b]$ are defined by

$$\begin{aligned}
\mathcal{U}\int_a^b f(x)\,dx &= \inf\left\{U(P, f) : P \text{ is a partition of } [a, b]\right\}, \\
\mathcal{L}\int_a^b f(x)\,dx &= \sup\left\{L(P, f) : P \text{ is a partition of } [a, b]\right\}.
\end{aligned}$$

The definition of upper and lower integrals thus requires that $\mathcal{U}\int_a^b f(x)\,dx$ be the infimum of the values of $U(P,f)$ and that $\mathcal{L}\int_a^b f(x)\,dx$ be the supremum of the values of $L(P,f)$ as P ranges over all possible partitions of the interval $[a,b]$.

Definition

A bounded function $f\colon [a,b] \to \mathbb{R}$ on a closed bounded interval $[a,b]$ is said to be *Riemann-integrable* (or *Darboux-integrable*) on $[a,b]$ if

$$\mathcal{U}\int_a^b f(x)\,dx = \mathcal{L}\int_a^b f(x)\,dx,$$

in which case the *Riemann integral* $\int_a^b f(x)\,dx$ (or *Darboux integral*) of f on $[a,b]$ is defined to be the common value of $\mathcal{U}\int_a^b f(x)\,dx$ and $\mathcal{L}\int_a^b f(x)\,dx$.

When $a > b$ we define

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

for all Riemann-integrable functions f on $[b, a]$. We set $\int_a^b f(x)\,dx = 0$ when $b = a$.

Any continuous real-valued function defined over a closed bounded interval is Riemann-integrable on that interval. This result can be proved without difficulty on applying the result that any continuous real-valued function defined over a closed bounded interval is uniformly continuous on that interval (see Theorem 5.11).

We now state without proof several results that follow as consequences of the definition of the Riemann integral. The proofs of these results are straightforward applications of the basic principles and standard proof techniques of real analysis.

Let $f\colon [a,b] \to \mathbb{R}$ and $g\colon [a,b] \to \mathbb{R}$ be bounded Riemann-integrable functions on a closed bounded interval $[a,b]$, where a and b are real numbers satisfying $a \leq b$. Then the functions $f+g$ and $f-g$ are Riemann-integrable on $[a,b]$, and moreover

$$\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx,$$

and

$$\int_a^b (f(x)-g(x))\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx.$$

Also

$$\int_a^b c\, f(x)\, dx = c \int_a^b f(x)\, dx,$$

for all real numbers c, and

$$\left| \int_a^b f(x)\, dx \right| \leq \int_a^b |f(x)|\, dx,$$

where $|f|\colon [a,b] \to \mathbb{R}$ is the function on $[a,b]$ defined such that $|f|(x) = |f(x)|$ for all $x \in [a,b]$. Moreover

$$\int_a^b f(x)\, dx = \int_a^s f(x)\, dx + \int_s^b f(x)\, dx.$$

for all real numbers s satisfying $a \leq s \leq b$. And if Riemann-integrable functions $f\colon [a,b] \to \mathbb{R}$ and $g\colon [a,b] \to \mathbb{R}$ satisfy $f(x) \leq g(x)$ for all $x \in [a,b]$, then

$$\int_a^b f(x)\, dx \leq \int_a^b g(x)\, dx.$$

11.2. Multiple Integrals of Bounded Continuous Functions

We consider multiple integrals involving continuous real-valued functions of several real variables over regions that are products of closed bounded intervals. Any subset of n-dimensional Euclidean space $\mathbb{R}^n$ that is a product of closed bounded intervals is a closed bounded set in $\mathbb{R}^n$. It follows from the Extreme Value Theorem (Theorem 5.10) that any continuous real-valued function on a product of closed bounded intervals is necessarily bounded on that product of intervals. It is also uniformly continuous on that product of intervals (see Theorem 5.11)

Proposition 11.1

Let n be an integer greater than 1, let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be real numbers, where $a_i < b_i$ for $i = 1, 2, \ldots, n$, let $f\colon [a_1, b_1] \times \cdots \times [a_n, b_n] \to \mathbb{R}$ be a continuous real-valued function, and let

$$g(x_1, x_2, \ldots, x_{n-1}) = \int_{a_n}^{b_n} f(x_1, x_2, \ldots, x_{n-1}, t)\, dt.$$

for all $(n-1)$-tuples $(x_1, x_2, \ldots, x_{n-1})$ of real numbers satisfying $a_i \leq x_i \leq b_i$ for $i = 1, 2, \ldots, n-1$. Then the function

$$g\colon [a_1, b_1] \times [a_2, b_2] \cdots \times [a_{n-1}, b_{n-1}] \to \mathbb{R}$$

is continuous.

Proof

Let some positive real number ε be given, and let ε_0 be chosen so that $0 < (b_n - a_n)\varepsilon_0 < \varepsilon$. The function f is uniformly continuous on $[a_1, b_1] \times [a_2, b_2] \cdots \times [a_n, b_n]$ (see Theorem 5.11). Therefore there exists some positive real number δ such that

$$|f(x_1, x_2, \ldots, x_{n-1}, t) - f(u_1, u_2, \ldots, u_{n-1}, t)| < \varepsilon_0$$

for all real numbers $x_1, x_2, \ldots, x_{n-1}$, $u_1, u_2, \ldots, u_{n-1}$ and t satisfying $a_i \leq x_i \leq b_i$, $a_i \leq u_i < b_i$ and $|x_i - u_i| < \delta$ for $i = 1, 2, \ldots, n-1$ and $a_n \leq t \leq b_n$. Consequently

$$\begin{aligned}
&|g(x_1, x_2, \ldots, x_{n-1}) - g(u_1, u_2, \ldots, u_{n-1})| \\
&\qquad = \left| \int_{a_n}^{b_n} (f(x_1, x_2, \ldots, x_{n-1}, t) - f(u_1, u_2, \ldots, u_{n-1}, t))\, dt \right| \\
&\qquad \leq \int_{a_n}^{b_n} |f(x_1, x_2, \ldots, x_{n-1}, t) - f(u_1, u_2, \ldots, u_{n-1}, t)|\, dt \\
&\qquad \leq \varepsilon_0 (b_n - a_n) < \varepsilon
\end{aligned}$$

whenever $a_i \leq x_i \leq b_i$, $a_i \leq u_i < b_i$ and $|x_i - u_i| < \delta$ for $i = 1, 2, \ldots, n-1$. The result follows. ∎

Proposition 11.1 ensures that, given a continuous real-valued function $f\colon [a_1, b_1] \times \cdots \times [a_n, b_n] \to \mathbb{R}$, where $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ are real numbers and $a_i < b_i$ for $i = 1, 2, \ldots, n$, there is a well-defined multiple integral

$$\int_{x_n=a_n}^{b_n} \cdots \int_{x_2=a_2}^{b_2} \int_{x_1=a_1}^{b_1} f(x_1, x_2, \ldots, x_n)\, dx_1\, dx_2 \cdots dx_n,$$

in which, at each stage of evaluation, the integrand is a continuous function of its arguments. To evaluate this integral, one integrates first with respect to x_1, then with respect to x_2, and so on, finally integrating with respect to x_n.

In fact, if the function f is continuous, the order of evaluation of the integrals with respect to the individual variables does not affect the value of the multiple integral. We prove this first for continuous functions of two variables.

Theorem 11.2

Let $f\colon [a_1, b_1] \times [a_2, b_2] \to \mathbb{R}$ be a continuous real-valued function on the closed rectangle $[a_1, b_1] \times [a_2, b_2]$. Then

$$\int_{a_2}^{b_2} \left(\int_{a_1}^{b_1} f(x, y)\, dx \right) dy = \int_{a_1}^{b_1} \left(\int_{a_2}^{b_2} f(x, y)\, dy \right) dx.$$

Proof

The function $f\colon [a_1, b_1] \times [a_2, b_2] \to \mathbb{R}$ is continuous, and is therefore uniformly continuous on $[a_1, b_1] \times [a_2, b_2]$ (see Theorem 5.11). Let some positive real number ε be given. It follows from the uniform continuity of the function f that there exists some positive real number δ with the property that

$$|f(x, y) - f(u, v)| < \varepsilon$$

for all $x, u \in [a_1, b_1]$ and $y, v \in [a_2, b_2]$ satisfying $|x - u| < \delta$ and $|y - v| < \delta$.

Let P be a partition of $[a_1, b_1]$, and let Q be a partition of $[a_2, b_2]$, where

$$P = \{u_0, u_1, \ldots, u_p\}, \quad Q = \{v_0, v_1, \ldots, v_q\},$$

$$a_1 = u_0 < u_1 < \cdots < u_p = b_1, \quad a_2 = v_0 < v_1 < \cdots < v_q = b_2,$$

$u_j - u_{j-1} < \delta$ for $j = 1, 2, \ldots, p$ and $v_k - v_{k-1} < \delta$ for $k = 1, 2, \ldots, q$. Then

$$|f(x, y) - f(u_j, v_k)| < \varepsilon$$

whenever $u_{j-1} \leq x \leq u_j$ for some integer j between 1 and p and $v_{k-1} \leq y \leq v_k$ for some integer k between 1 and q.

Now

$$\int_{a_2}^{b_2} \left(\int_{a_1}^{b_1} f(x,y)\,dx \right)\,dy = \sum_{k=1}^{q} \sum_{j=1}^{p} \int_{v_{k-1}}^{v_k} \left(\int_{u_{j-1}}^{u_j} f(x,y)\,dx \right)\,dy.$$

Moreover

$$\int_{u_{j-1}}^{u_j} f(x,y)\,dx \leq \Big(f(u_j,v_k) + \varepsilon \Big)(u_j - u_{j-1})$$

for all $y \in [v_{k-1}, v_k]$, and therefore

$$\int_{v_{k-1}}^{v_k} \left(\int_{u_{j-1}}^{u_j} f(x,y)\,dx \right)\,dy \leq \Big(f(u_j,v_k) + \varepsilon \Big)(v_k - v_{k-1})(u_j - u_{j-1})$$

for all integers j between 1 and p and integers k between 1 and q.

It follows that

$$\begin{aligned}
&\int_{a_2}^{b_2} \left(\int_{a_1}^{b_1} f(x,y)\,dx \right)\,dy \\
&\quad \leq \sum_{k=1}^{q} \sum_{j=1}^{p} \Big(f(u_j, v_k) + \varepsilon \Big) (v_k - v_{k-1})(u_j - u_{j-1}) \\
&\quad = S + \varepsilon (b_1 - a_1)(b_2 - a_2),
\end{aligned}$$

where

$$S = \sum_{k=1}^{q} \sum_{j=1}^{p} f(u_j, v_k)(v_k - v_{k-1})(u_j - u_{j-1}).$$

Similarly

$$\begin{aligned}
&\int_{a_2}^{b_2} \left(\int_{a_1}^{b_1} f(x,y)\,dx \right)\,dy \\
&\quad \geq \sum_{k=1}^{q} \sum_{j=1}^{p} \Big(f(u_j, v_k) - \varepsilon \Big)(v_k - v_{k-1})(u_j - u_{j-1}) \\
&\quad = S - \varepsilon(b_1 - a_1)(b_2 - a_2).
\end{aligned}$$

Thus

$$\left| \int_{a_2}^{b_2} \left(\int_{a_1}^{b_1} f(x,y)\,dx \right)\,dy - S \right| \leq \varepsilon(b_1 - a_1)(b_2 - a_2).$$

On interchanging the roles of the variables x and y, we conclude similarly that

$$\left| \int_{a_1}^{b_1} \left(\int_{a_2}^{b_2} f(x,y)\,dy \right)\,dx - S \right| \leq \varepsilon(b_2 - a_2)(b_1 - a_1).$$

It follows that

$$\begin{aligned} \left| \int_{a_2}^{b_2} \left(\int_{a_1}^{b_1} f(x,y)\,dx \right)\,dy - \int_{a_1}^{b_1} \left(\int_{a_2}^{b_2} f(x,y)\,dy \right)\,dx \right| & \\ \leq \quad 2\varepsilon(b_1 - a_1)(b_2 - a_2). & \end{aligned}$$

Moreover the inequality just obtained must hold for every positive real number ε, no matter how small the value of ε. It follows that

$$\int_{a_2}^{b_2} \left(\int_{a_1}^{b_1} f(x,y)\,dx \right)\,dy = \int_{a_1}^{b_1} \left(\int_{a_2}^{b_2} f(x,y)\,dy \right)\,dx,$$

as required. $\blacksquare$

Now let us consider a multiple integral involving a continuous function of three real variables. Let

$$f : [a_1, b_1] \times [a_2, b_2] \times [a_3, b_3] \to \mathbb{R}$$

be a continuous real-valued function, where a_1, a_2, a_3, b_1, b_2 and b_3 are real numbers satisfying $a_1 < b_1$, $a_2 < b_2$ and $a_3 < b_3$. It follows from Theorem 11.2 that

$$\int_{a_1}^{b_1} \int_{a_2}^{b_2} f(x_1, x_2, x_3)\, dx_2\, dx_1 = \int_{a_2}^{b_2} \int_{a_1}^{b_1} f(x_1, x_2, x_3)\, dx_1\, dx_2$$

for all real numbers x_3 satisfying $a_3 < x_3 < b_3$. It follows that

$$\begin{aligned}\int_{a_3}^{b_3} \int_{a_1}^{b_1} \int_{a_2}^{b_2} & f(x_1, x_2, x_3)\, dx_2\, dx_1\, dx_3 \\ &= \int_{a_3}^{b_3} \int_{a_2}^{b_2} \int_{a_1}^{b_1} f(x_1, x_2, x_3)\, dx_1\, dx_2\, dx_3.\end{aligned}$$

Also it follows from Proposition 11.1 that the function sending (x_2, x_3) to

$$\int_{a_1}^{b_1} f(x_1, x_2, x_3)\, dx_1$$

for all $(x_2, x_3) \in [a_2, b_2] \times [a_3, b_3]$ is a continuous function of (x_2, x_3). It then follows from Theorem 11.2 that

$$\begin{aligned} &\int_{a_2}^{b_2} \int_{a_3}^{b_3} \int_{a_1}^{b_1} f(x_1, x_2, x_3)\, dx_1\, dx_3\, dx_2 \\ &\qquad = \int_{a_3}^{b_3} \int_{a_2}^{b_2} \int_{a_1}^{b_1} f(x_1, x_2, x_3)\, dx_1\, dx_2\, dx_3. \end{aligned}$$

Repeated applications of these results establish that the value of the repeated integral with respect to the real variables x_1, x_2 and x_3 is independent of the order in which the successive integrations are performed.

Corresponding results hold for integration of continuous real-valued functions of four or more real variables. In general, if the integrand is a continuous real-valued function of n real variables, and if this function is integrated over a product of n closed bounded intervals, by repeated integration, then the value of the integral is independent of the order in which the integrals are performed.

11.3. A Counterexample involving an Unbounded Function

Example

Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be defined such that

$$f(x,y) = \begin{cases} \dfrac{4xy(x^2-y^2)}{(x^2+y^2)^3} & \text{if } (x,y) \neq (0,0); \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Set $u = x^2 + y^2$. Then

$$f(x,y) = \frac{2x(2x^2-u)}{u^3}\frac{\partial u}{\partial y},$$

and therefore, when $x \neq 0$,

$$\begin{aligned}
\int_{y=0}^{1} f(x,y)\,dy &= \int_{u=x^2}^{x^2+1} \left(\frac{4x^3}{u^3} - \frac{2x}{u^2} \right) du \\
&= \left[-\frac{2x^3}{u^2} + \frac{2x}{u} \right]_{u=x^2}^{x^2+1} \\
&= -\frac{2x^3}{(x^2+1)^2} + \frac{2x}{x^2+1} \\
&= \frac{2x}{(x^2+1)^2}
\end{aligned}$$

It follows that

$$\begin{aligned}
\int_{x=0}^{1} \left(\int_{y=0}^{1} f(x,y)\,dy \right)\,dx &= \int_{x=0}^{1} \frac{2x}{(x^2+1)^2}\,dx \\
&= \left[-\frac{1}{x^2+1} \right]_0^1 = \frac{1}{2}.
\end{aligned}$$

Now $f(y,x) = -f(x,y)$ for all x and y. Interchanging x and y in the above evaluation, we find that

$$\begin{aligned}
\int_{y=0}^{1} \left(\int_{x=0}^{1} f(x,y)\,dx \right)\,dy &= \int_{x=0}^{1} \left(\int_{y=0}^{1} f(y,x)\,dy \right)\,dx \\
&= -\int_{x=0}^{1} \left(\int_{y=0}^{1} f(x,y)\,dy \right)\,dx \\
&= -\frac{1}{2}.
\end{aligned}$$

Thus

$$\int_{x=0}^{1} \left(\int_{y=0}^{1} f(x,y)\,dy \right)\,dx \neq \int_{y=0}^{1} \left(\int_{x=0}^{1} f(x,y)\,dx \right)\,dy.$$

when

$$f(x,y) = \frac{4xy(x^2-y^2)}{(x^2+y^2)^3}$$

for all $(x,y) \in \mathbb{R}^2$ distinct from $(0,0)$. Note that, in this case $f(2t,t) \to +\infty$ as $t \to 0^+$, and $f(t,2t) \to -\infty$ as $t \to 0^-$. Thus the function f is not continuous at $(0,0)$ and does not remain bounded as $(x,y) \to (0,0)$.