MAU23203: Analysis in Several Real Variables
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Disquisition VII: An Example of Differentiation from First Principles

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Example Let $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the function defined such that

$$\varphi \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} e^x \cos y \\ e^x \sin y \end{pmatrix}$$

for all $(x, y) \in \mathbb{R}^2$. Then

$$\begin{aligned}
\varphi \begin{pmatrix} x+h \\ y+k \end{pmatrix} &= \begin{pmatrix} e^{x+h} \cos(y+k) \\ e^{x+h} \sin(y+k) \end{pmatrix} \\
&= \begin{pmatrix} e^x\, e^h \cos y \cos k - e^x e^h \sin y \sin k, \\ e^x\, e^h \sin y \cos k + e^x e^h \cos y \sin k \end{pmatrix} \\
&= e^h \cos k \begin{pmatrix} e^x \cos y \\ e^x \sin y \end{pmatrix} + e^h \sin k \begin{pmatrix} -e^x \sin y \\ e^x \cos y \end{pmatrix} \\
&= \begin{pmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{pmatrix} \begin{pmatrix} e^h \cos k \\ e^h \sin k \end{pmatrix} \\
&= \begin{pmatrix} e^x \cos y \\ e^x \sin y \end{pmatrix} + \begin{pmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{pmatrix} \begin{pmatrix} h \\ k \end{pmatrix} \\
&\qquad + \mathbf{E}(x, y, h, k),
\end{aligned}$$

where

$$\mathbf{E}(x, y, h, k) = \begin{pmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{pmatrix} \begin{pmatrix} e^h \cos k - 1 - h \\ e^h \sin k - k \end{pmatrix}.$$

Now

$$e^h \cos k - 1 - h = (e^h - 1 - h) - e^h(1 - \cos k).$$

Moreover, using methods of one variable calculus, such as l'Hôpital's Rule, one can verify that

$$\lim_{h\to 0} \frac{e^h - 1 - h}{h^2} = \frac{1}{2} \quad \text{and} \quad \lim_{k\to 0} \frac{1 - \cos k}{k^2} = \frac{1}{2}$$

It follows easily from these results that there exist positive real numbers K_1 and δ_1 such that

$$|e^h \cos k - 1 - h| \le K_1(h^2 + k^2)$$

whenever $\sqrt{h^2 + k^2} < \delta_1$. Next we note that

$$e^h \sin k - k = (e^h - 1)\sin k + \sin k - k,$$

Moreover, using methods of one variable calculus, such as l'Hôpital's Rule, one can verify that

$$\lim_{h\to 0} \frac{e^h - 1}{h} = 1, \quad \lim_{k\to 0} \frac{\sin k}{k} = 1, \quad \lim_{k\to 0} \frac{\sin k - k}{k^2} = 0.$$

Moreover $|hk| \le \frac{1}{2}(h^2 + k^2)$. It follows easily from these results that there exist positive real numbers K_2 and δ_2 such that

$$|e^h \sin k - k| \le K_2(h^2 + k^2)$$

whenever $\sqrt{h^2 + k^2} < \delta_2$. And also there exists a positive constant L, depending on x and y, but not on h and k, such that

$$|\mathbf{E}(x, y, h, k)| \le L\sqrt{(e^h \cos k - 1 - h)^2 + (e^h \sin k - k)^2}$$

for all real numbers h and k. It follows that there exist positive constant K and δ, depending on x and y but not on h and k, such that

$$|\mathbf{E}(x, y, h, k)| \le K(h^2 + k^2)$$

whenever $\sqrt{h^2 + k^2} < \delta$. It follows from this that

$$\frac{1}{\sqrt{h^2 + k^2}}|\mathbf{E}(x, y, h, k)| \le K\sqrt{h^2 + k^2}$$

whenever $\sqrt{h^2 + k^2} < \delta$, and consequently

$$\lim_{(h,k)\to(0,0)} \frac{1}{\sqrt{h^2 + k^2}}|\mathbf{E}(x, y, h, k)| = 0.$$

Thus

$$\lim_{(h,k)\to(0,0)} \frac{1}{\sqrt{h^2+k^2}} \left(\varphi \begin{pmatrix} x+h \\ y+k \end{pmatrix} - \varphi \begin{pmatrix} x \\ y \end{pmatrix} \right.$$

$$\left. - \begin{pmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{pmatrix} \begin{pmatrix} h \\ k \end{pmatrix} \right) = 0.$$

This establishes that the function φ is differentiable at (x, y) and that the derivative $(D\varphi)_{(x,y)}$ is the linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ represented by the matrix

$$\begin{pmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{pmatrix}.$$