

MAU23203: Analysis in Several Real Variables  
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Disquisition I: Convergent Subsequence  
Examples

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**An Example concerning Convergent Subsequences of a Bounded Infinite Sequence**

We recall that an infinite sequence  $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \dots$  of points in  $n$ -dimensional Euclidean space  $\mathbb{R}^n$  is said to be *bounded* if there exists some constant  $K$  such that  $|\mathbf{x}_j| \leq K$  for all  $j$ .

The multidimensional Bolzano-Weierstrass Theorem asserts that every bounded infinite sequence of points in a Euclidean space has a convergent subsequence.

The basis strategy for proving this theorem is exemplified in the following 3-dimensional example.

**Example** Let

$$(x_j, y_j, z_j) = \left( \sin(\pi\sqrt{j}), (-1)^j, \cos\left(\frac{2\pi \log j}{\log 2}\right) \right)$$

for  $j = 1, 2, 3, \dots$ . This infinite sequence of points in  $\mathbb{R}^3$  is bounded, because the components of its members all take values between  $-1$  and  $1$ . Moreover  $x_j = 0$  whenever  $j$  is the square of a positive integer,  $y_j = 1$  whenever  $j$  is even and  $z_j = 1$  whenever  $j$  is a power of two.

The infinite sequence  $x_1, x_2, x_3, \dots$  has a convergent subsequence

$$x_1, x_4, x_9, x_{16}, x_{25}, \dots$$

which includes those  $x_j$  for which  $j$  is the square of a positive integer. The corresponding subsequence  $y_1, y_4, y_9, \dots$  of  $y_1, y_2, y_3, \dots$  is not convergent, because its values alternate between 1 and  $-1$ . However this subsequence is bounded, and we can extract from this sequence a convergent subsequence

$$y_4, y_{16}, y_{36}, y_{64}, y_{100}, \dots$$

which includes those  $x_j$  for which  $j$  is the square of an even positive integer.

The subsequence

$$x_4, x_{16}, x_{36}, y_{64}, y_{100}, \dots$$

is also convergent, because it is a subsequence of a convergent subsequence. However the corresponding subsequence

$$z_4, z_{16}, z_{36}, z_{64}, z_{100}, \dots$$

does not converge. (Indeed  $z_j = 1$  when  $j$  is an even power of 2, but  $z_j = \cos(2\pi \log(9)/\log(2))$  when  $j = 9 \times 2^{2p}$  for some positive integer  $p$ .) However this subsequence is bounded, and we can extract from it a convergent subsequence

$$z_4, z_{16}, z_{64}, z_{256}, z_{1024}, \dots$$

which includes those  $x_j$  for which  $j$  is equal to two raised to the power of an even positive integer. Then the first, second and third components of the following subsequence

$$(x_4, y_4, z_4), (x_{16}, y_{16}, z_{16}), (x_{64}, y_{64}, z_{64}), (x_{256}, y_{256}, z_{256}), \dots$$

of the original sequence of points in  $\mathbb{R}^3$  converge, and consequently this sequence is a convergent subsequence of the given infinite sequence of points in  $\mathbb{R}^3$ .

**Example** Let

$$x_j = \begin{cases} 1 & \text{if } j = 4k \text{ for some integer } k \\ 0 & \text{if } j = 4k + 1 \text{ for some integer } k \\ -1 & \text{if } j = 4k + 2 \text{ for some integer } k \\ 0 & \text{if } j = 4k + 3 \text{ for some integer } k \end{cases}$$

and

$$y_j = \begin{cases} 0 & \text{if } j = 4k \text{ for some integer } k, \\ 1 & \text{if } j = 4k + 1 \text{ for some integer } k, \\ 0 & \text{if } j = 4k + 2 \text{ for some integer } k, \\ -1 & \text{if } j = 4k + 3 \text{ for some integer } k, \end{cases},$$

and let  $\mathbf{u}_j = (x_j, y_j)$  for  $j = 1, 2, 3, 4, \dots$ . Then the first components  $x_j$  for which the index  $j$  is odd constitute a convergent sequence  $x_1, x_3, x_5, x_7, \dots$  of real numbers, and the second components  $y_j$  for which the index  $j$  is even also constitute a convergent sequence  $y_2, y_4, y_6, y_8, \dots$  of real numbers.

However one would not obtain a convergent subsequence of  $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots$  simply by selecting those indices  $j$  for which  $x_j$  is in the convergent subsequence  $x_1, x_3, x_5, \dots$  and  $y_j$  is in the convergent subsequence  $y_2, y_4, y_6, \dots$ , because there are no values of the index  $j$  for which  $x_j$  and  $y_j$  both belong to the respective subsequences. However the one-dimensional Bolzano-Weierstrass Theorem (Theorem 3.3) guarantees that there is a convergent subsequence of  $y_1, y_3, y_5, y_7, \dots$ , and indeed  $y_1, y_5, y_9, y_{13}, \dots$  is such a convergent subsequence. This yields a convergent subsequence  $\mathbf{u}_1, \mathbf{u}_5, \mathbf{u}_9, \mathbf{u}_{13}, \dots$  of the given bounded infinite sequence of points in  $\mathbb{R}^2$ .