# Module MAU23203: Analysis in Several Real Variables Michaelmas Term 2023 Section 3: Convergence in Euclidean Spaces

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## **3** Convergence in Euclidean Spaces

## 3.1 Convergence of Infinite Sequences of Real Numbers

An *infinite sequence*  $x_1, x_2, x_3, \ldots$  of real numbers associates to each positive integer j a corresponding real number  $x_j$ .

**Definition** An infinite sequence  $x_1, x_2, x_3, \ldots$  of real numbers is said to *converge* to some real number p if and only if the following criterion is satisfied:

given any strictly positive real number  $\varepsilon$ , there exists some positive integer N such that  $|x_j - p| < \varepsilon$  for all positive integers j satisfying  $j \ge N$ .

If an infinite sequence  $x_1, x_2, x_3, \ldots$  of real numbers converges to some real number p, then p is said to be the *limit* of the sequence, and we can indicate the convergence of the infinite sequence to p by writing ' $x_j \to p$  as  $j \to +\infty$ ', or by writing ' $\lim_{j\to+\infty} x_j = p$ '.

Let x and p be real numbers, and let  $\varepsilon$  be a strictly positive real number. Then  $|x - p| < \varepsilon$  if and only if both  $x - p < \varepsilon$  and  $p - x < \varepsilon$ . It follows that  $|x - p| < \varepsilon$  if and only if  $p - \varepsilon < x < p + \varepsilon$ . The condition  $|x - p| < \varepsilon$  essentially requires that the value of the real number x should agree with p to within an error of at most  $\varepsilon$ . An infinite sequence  $x_1, x_2, x_3, \ldots$  of real numbers converges to some real number p if and only if, given any positive real number  $\varepsilon$ , there exists some positive integer N such that  $p - \varepsilon < x_j < p + \varepsilon$  for all positive integers j satisfying  $j \ge N$ .

**Definition** We say that an infinite sequence  $x_1, x_2, x_3, \ldots$  of real numbers is bounded above if there exists some real number B such that  $x_j \leq B$  for all positive integers j. Similarly we say that this sequence is bounded below if there exists some real number A such that  $x_j \geq A$  for all positive integers j. A sequence is said to be bounded if it is bounded above and bounded below. Thus the sequence  $x_1, x_2, x_3, \ldots$  is bounded if and only if there exist real numbers A and B such that  $A \leq x_j \leq B$  for all positive integers j.

Lemma 3.1 Every convergent infinite sequence of real numbers is bounded.

**Proof** Let  $x_1, x_2, x_3, \ldots$  be an infinite sequence of real numbers that converges to some real number p. On applying the formal definition of convergence (with  $\varepsilon = 1$ ), we deduce the existence of some positive integer N such that  $p - 1 < x_j < p + 1$  for all  $j \ge N$ . But then  $A \le x_j \le B$  for all positive integers j, where A is the minimum of  $x_1, x_2, \ldots, x_{N-1}$  and p - 1, and B is the maximum of  $x_1, x_2, \ldots, x_{N-1}$  and p + 1.

### **3.2** Monotonic Sequences

An infinite sequence  $x_1, x_2, x_3, \ldots$  of real numbers is said to be *strictly increasing* if  $x_{j+1} > x_j$  for all positive integers j, *strictly decreasing* if  $x_{j+1} < x_j$  for all positive integers j, *non-decreasing* if  $x_{j+1} \ge x_j$  for all positive integers j, *non-increasing* if  $x_{j+1} \le x_j$  for all positive integers j. A sequence satisfying any one of these conditions is said to be *monotonic*; thus a monotonic sequence is either non-decreasing or non-increasing.

**Theorem 3.2** Any non-decreasing infinite sequence of real numbers that is bounded above is convergent. Similarly any non-increasing infinite sequence of real numbers that is bounded below is convergent.

**Proof** Let  $x_1, x_2, x_3, \ldots$  be a non-decreasing sequence of real numbers that is bounded above. It follows from the Least Upper Bound Principle that there exists a least upper bound p for the set  $\{x_j : j \in \mathbb{N}\}$ . We claim that the sequence converges to p.

Let some strictly positive real number  $\varepsilon$  be given. We must show that there exists some positive integer N such that  $|x_j - p| < \varepsilon$  whenever  $j \ge N$ . Now  $p - \varepsilon$  is not an upper bound for the set  $\{x_j : j \in \mathbb{N}\}$  (because p is the least upper bound), and therefore there must exist some positive integer N such that  $x_N > p - \varepsilon$ . But then  $p - \varepsilon < x_j \le p$  whenever  $j \ge N$ , since the sequence is non-decreasing and bounded above by the real number p. Thus  $|x_j - p| < \varepsilon$  whenever  $j \ge N$ . Therefore  $x_j \to p$  as  $j \to +\infty$ , as required.

Next we note that if an infinite sequence  $x_1, x_2, x_3, \ldots$  is non-increasing and bounded below then the sequence  $-x_1, -x_2, -x_3, \ldots$  is non-decreasing and bounded above, and is therefore convergent. It follows that the sequence  $x_1, x_2, x_3, \ldots$  is also convergent.

### **3.3** Subsequences of Sequences of Real Numbers

**Definition** Let  $x_1, x_2, x_3, \ldots$  be an infinite sequence of real numbers. A subsequence of this infinite sequence is a sequence of the form  $x_{j_1}, x_{j_2}, x_{j_3}, \ldots$  where  $j_1, j_2, j_3, \ldots$  is an infinite sequence of positive integers with

$$j_1 < j_2 < j_3 < \cdots$$

Let  $x_1, x_2, x_3, \ldots$  be an infinite sequence of real numbers. The following sequences are examples of subsequences of this sequence:—

$$x_1, x_3, x_5, x_7, \dots$$
  
 $x_1, x_4, x_9, x_{16}, \dots$ 

## 3.4 The Bolzano-Weierstrass Theorem in One Dimension

**Theorem 3.3 (Bolzano-Weierstrass for the Real Line)** Every bounded infinite sequence of real numbers has a convergent subsequence.

**Proof** Let some bounded infinite sequence  $x_1, x_2, x_3, \ldots$  of real numbers be given. We define a *peak index* to be a positive integer j with the property that  $x_j \ge x_k$  for all positive integers k satisfying  $k \ge j$ . Thus a positive integer j is a peak index if and only if the jth member of the infinite sequence  $x_1, x_2, x_3, \ldots$  is greater than or equal to all succeeding members of the sequence. Let S be the set consisting of all peak indices. Then

$$S = \{ j \in \mathbb{N} : x_j \ge x_k \text{ for all } k \ge j \}.$$

First let us suppose that the set of peak indices is infinite. Arrange the set S of peak indices in increasing order so that  $S = \{j_1, j_2, j_3, j_4, \ldots\}$ , where  $j_1 < j_2 < j_3 < j_4 < \cdots$ . It follows from the definition of peak indices that  $x_{j_1} \ge x_{j_2} \ge x_{j_3} \ge x_{j_4} \ge \cdots$ . Thus  $x_{j_1}, x_{j_2}, x_{j_3}, \ldots$  is a non-increasing subsequence of the given infinite sequence  $x_1, x_2, x_3, \ldots$ . This subsequence is bounded below (since the given infinite sequence is bounded). It follows from Theorem 3.2 that  $x_{j_1}, x_{j_2}, x_{j_3}, \ldots$  is a convergent subsequence of the given infinite sequence.

Now suppose that the set S of peak indices is finite. Choose a positive integer  $j_1$  which is greater than every peak index. Then  $j_1$  is not a peak index. Therefore there must exist some positive integer  $j_2$  satisfying  $j_2 > j_1$  such that  $x_{j_2} > x_{j_1}$ . Moreover  $j_2$  is not a peak index (because  $j_2$  is greater than  $j_1$  and  $j_1$  in turn is greater than every peak index). Therefore there must exist some positive integer  $j_3$  satisfying  $j_3 > j_2$  such that  $x_{j_3} > x_{j_2}$ . We can continue in this fashion to construct (by induction on j) a strictly increasing subsequence  $x_{j_1}, x_{j_2}, x_{j_3}, \ldots$  of our original sequence. This increasing subsequence is bounded above (since the original sequence is bounded) and thus is convergent, by Theorem 3.2. This completes the proof of the one-dimensional case of the Bolzano-Weierstrass Theorem.

## 3.5 Convergence of Sequences in Euclidean Spaces

**Definition** An infinite sequence  $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$  of points in  $\mathbb{R}^n$  is said to *converge* to a point  $\mathbf{p}$  if and only if, given any strictly positive real number  $\varepsilon$ , there exists some positive integer N such that  $|\mathbf{x}_j - \mathbf{p}| < \varepsilon$  whenever  $j \ge N$ .

Given a convergent infinite sequence  $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$  of points in  $\mathbb{R}^n$ , the point  $\mathbf{p}$  to which the sequence converges is referred to as the *limit* of the infinite sequence, and may be denoted by  $\lim_{i \to +\infty} \mathbf{x}_j$ .

**Lemma 3.4** Let  $\mathbf{p}$  be a point of  $\mathbb{R}^n$ , where  $\mathbf{p} = (p_1, p_2, \ldots, p_n)$ . Then an infinite sequence  $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$  of points in  $\mathbb{R}^n$  converges to  $\mathbf{p}$  if and only if the *i*th components of the elements of this sequence converge to  $p_i$  for  $i = 1, 2, \ldots, n$ .

**Proof** For each positive integer j, let  $(\mathbf{x}_j)_i$  denote the *i*th component of  $\mathbf{x}_j$ . Then  $|(\mathbf{x}_j)_i - p_i| \leq |\mathbf{x}_j - \mathbf{p}|$  for i = 1, 2, ..., n and for all positive integers j. It follows directly from the definition of convergence that if  $\mathbf{x}_j \to \mathbf{p}$  as  $j \to +\infty$  then  $(\mathbf{x}_j)_i \to p_i$  as  $j \to +\infty$ .

Conversely suppose that, for each integer *i* between 1 and n,  $(\mathbf{x}_j)_i \to p_i$  as  $j \to +\infty$ . Let some positive real number  $\varepsilon$  be given. Then there exist positive integers  $N_1, N_2, \ldots, N_n$  such that  $|(\mathbf{x}_j)_i - p_i| < \varepsilon/\sqrt{n}$  whenever  $j \ge N_i$ . Let N be the maximum of  $N_1, N_2, \ldots, N_n$ . If  $j \ge N$  then  $j \ge N_i$  for  $i = 1, 2, \ldots, n$ , and therefore

$$|\mathbf{x}_j - \mathbf{p}|^2 = \sum_{i=1}^n ((\mathbf{x}_j)_i - p_i)^2 < n \left(\frac{\varepsilon}{\sqrt{n}}\right)^2 = \varepsilon^2.$$

Thus  $\mathbf{x}_j \to \mathbf{p}$  as  $j \to +\infty$ , as required.

## 3.6 The Multidimensional Bolzano-Weierstrass Theorem

#### Theorem 3.5 (Multidimensional Bolzano-Weierstrass Theorem)

Every bounded sequence of points in a Euclidean space has a convergent subsequence.

**Proof** The theorem is proved by induction on the dimension n of the space  $\mathbb{R}^n$  within which the points reside. When n = 1, the required result is the one-dimensional case of the Bolzano-Weierstrass Theorem, and the result has already been established in this case (see Theorem 3.3).

When n > 1, the result is proved in dimension n assuming the result in dimensions n - 1 and 1. Consequently the result is established successively in dimensions 2, 3, 4, ..., and therefore is valid for bounded sequences in  $\mathbb{R}^n$  for all positive integers n.

It has been shown that every bounded infinite sequence of real numbers has a convergent subsequence (Theorem 3.3). Let n be an integer greater than

one, and suppose, as an induction hypothesis, that, in cases where n > 2, all bounded sequences of points in  $\mathbb{R}^{n-1}$  have convergent subsequences. Let  $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$  be a bounded infinite sequence in  $\mathbb{R}^n$  and, for each positive integer j, let  $\mathbf{s}_j$  denote the point of  $\mathbb{R}^{n-1}$  whose *i*th component is equal to the *i*th component  $x_{j,i}$  of  $\mathbf{x}_j$  for each integer i between 1 and n-1.

Let some strictly positive real number  $\varepsilon$  be given. Now the infinite sequence

 $\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3, \dots$ 

of points of  $\mathbb{R}^{n-1}$  is a bounded infinite sequence. In the case when n = 2 we can apply the one-dimensional Bolzano-Weierstrass Theorem (Theorem 3.3) to conclude that this sequence of real numbers has a convergent subsequence. In cases where n > 2, we are supposing as our induction hypothesis that any bounded sequence in  $\mathbb{R}^{n-1}$  has a convergent subsequence. Thus, assuming this induction hypothesis in cases where n > 2, we can conclude, in all cases with n > 1, that the bounded infinite sequence  $\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3, \ldots$  of points in  $\mathbb{R}^{n-1}$  has a convergent subsequence be

$$\mathbf{s}_{m_1}, \mathbf{s}_{m_2}, \mathbf{s}_{m_3}, \ldots,$$

where  $m_1, m_2, m_3, \ldots$  is a strictly increasing infinite sequence of positive integers, and let  $\mathbf{q} = \lim_{j \to +\infty} \mathbf{s}_{m_j}$ . There then exists some positive integer L such that

$$|\mathbf{s}_{m_j} - \mathbf{q}| < rac{1}{2} arepsilon$$

for all positive integers j for which  $m_j \ge L$ . (Indeed the definition of convergence ensures the existence of a positive integer N that is large enough to ensure that  $|\mathbf{s}_{m_j} - \mathbf{q}| < \frac{1}{2}\varepsilon$  whenever  $j \ge N$ . Taking  $L = m_N$  then ensures that  $j \ge N$  whenever  $m_j \ge L$ .)

Let  $t_j$  denote the *n*th component of the point  $\mathbf{x}_j$  of  $\mathbb{R}^n$  for each positive integer j. The one-dimensional Bolzano-Weierstrass Theorem ensures that the bounded infinite sequence

$$t_{m_1}, t_{m_2}, t_{m_3}, \ldots$$

of real numbers has a convergent subsequence. It follows that there is a strictly increasing infinite sequence  $k_1, k_2, k_3, \ldots$  of positive integers, where each  $k_j$  is equal to one of the positive integers  $m_1, m_2, m_3, \ldots$ , such that the infinite sequence

$$t_{k_1}, t_{k_2}, t_{k_3}, \ldots$$

is convergent. Let  $r = \lim_{j \to +\infty} t_{k_j}$ . There then exists some positive integer M such that  $M \ge L$  and

$$|t_{k_i} - r| < \frac{1}{2}\varepsilon$$

for all positive integers j for which  $k_j \ge M$ . It follows that if  $k_j \ge M$  then

$$|\mathbf{s}_{k_j} - \mathbf{q}| < \frac{1}{2}\varepsilon$$
 and  $|t_{k_j} - r| < \frac{1}{2}\varepsilon$ .

Now there is a point  $\mathbf{p}$  of  $\mathbb{R}^n$ , where  $\mathbf{p} = (p_1, p_2, \dots, p_n)$ , determined so that the *i*th components of the point  $\mathbf{p}$  of  $\mathbb{R}^n$  is equal to the *i*th component of the point  $\mathbf{q}$  of  $\mathbb{R}^{n-1}$  for each integer *i* between 1 and n-1 and also the *n*th component of the point  $\mathbf{p}$  is equal to the real number *r*.

Also it follows from the definition of the Euclidean norm that

$$|\mathbf{x}_{k_j} - \mathbf{p}|^2 = |\mathbf{s}_{k_j} - \mathbf{q}|^2 + |t_{k_j} - r|^2 < \frac{1}{2}\varepsilon^2$$

whenever  $k_j \geq M$ . But then  $|\mathbf{x}_{k_j} - \mathbf{p}| < \varepsilon$  for all positive integers j for which  $k_j \geq M$ . It follows that  $\lim_{j \to +\infty} \mathbf{x}_{k_j} = \mathbf{p}$ . We conclude therefore that the bounded infinite sequence  $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$  does indeed have a convergent subsequence. This completes the proof of the Bolzano-Weierstrass Theorem in dimension n for all positive integers n.

### 3.7 Cauchy Sequences in Euclidean Spaces

**Definition** An infinite sequence  $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$  of points of *n*-dimensional Euclidean space  $\mathbb{R}^n$  is said to be a *Cauchy sequence* if, given any strictly positive real number  $\varepsilon$ , there exists some positive integer N such that  $|\mathbf{x}_j - \mathbf{x}_k| < \varepsilon$  for all positive integers j and k satisfying  $j \geq N$  and  $k \geq N$ .

**Lemma 3.6** Every Cauchy sequence of points of n-dimensional Euclidean space  $\mathbb{R}^n$  is bounded.

**Proof** Let  $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$  be a Cauchy sequence of points in  $\mathbb{R}^n$ . Then there exists some positive integer N such that  $|\mathbf{x}_j - \mathbf{x}_k| < 1$  whenever  $j \geq N$  and  $k \geq N$ . In particular,  $|\mathbf{x}_j| \leq |\mathbf{x}_N| + 1$  whenever  $j \geq N$ . Therefore  $|\mathbf{x}_j| \leq R$  for all positive integers j, where R is the maximum of the real numbers  $|\mathbf{x}_1|, |\mathbf{x}_2|, \ldots, |\mathbf{x}_{N-1}|$  and  $|\mathbf{x}_N| + 1$ . Thus the sequence is bounded, as required.

**Theorem 3.7 (Cauchy's Criterion for Convergence)** An infinite sequence of points of n-dimensional Euclidean space  $\mathbb{R}^n$  is convergent if and only if it is a Cauchy sequence.

**Proof** First we show that convergent sequences in  $\mathbb{R}^n$  are Cauchy sequences. Let  $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$  be a convergent sequence of points in  $\mathbb{R}^n$ , and let  $\mathbf{p} = \lim_{j \to +\infty} \mathbf{x}_j$ . Let some strictly positive real number  $\varepsilon$  be given. Then there exists some positive integer N such that  $|\mathbf{x}_j - \mathbf{p}| < \frac{1}{2}\varepsilon$  for all  $j \ge N$ . Thus if  $j \ge N$  and  $k \ge N$  then  $|\mathbf{x}_j - \mathbf{p}| < \frac{1}{2}\varepsilon$  and  $|\mathbf{x}_k - \mathbf{p}| < \frac{1}{2}\varepsilon$ , and hence

$$|\mathbf{x}_j - \mathbf{x}_k| = |(\mathbf{x}_j - \mathbf{p}) - (\mathbf{x}_k - \mathbf{p})| \le |\mathbf{x}_j - \mathbf{p}| + |\mathbf{x}_k - \mathbf{p}| < \varepsilon.$$

Thus the sequence  $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$  is a Cauchy sequence.

Conversely we must show that any Cauchy sequence  $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$  in  $\mathbb{R}^n$  is convergent. Now Cauchy sequences are bounded, by Lemma 3.6. The sequence  $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$  therefore has a convergent subsequence  $\mathbf{x}_{k_1}, \mathbf{x}_{k_2}, \mathbf{x}_{k_3}, \ldots$ , by the multidimensional Bolzano-Weierstrass Theorem (Theorem 3.5). Let  $\mathbf{p} = \lim_{j \to +\infty} \mathbf{x}_{k_j}$ . We claim that the sequence  $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$  itself converges to  $\mathbf{p}$ .

Let some strictly positive real number  $\varepsilon$  be given. Then there exists some positive integer N such that  $|\mathbf{x}_j - \mathbf{x}_k| < \frac{1}{2}\varepsilon$  whenever  $j \ge N$  and  $k \ge N$ (since the sequence is a Cauchy sequence). Let m be chosen large enough to ensure that  $k_m \ge N$  and  $|\mathbf{x}_{k_m} - \mathbf{p}| < \frac{1}{2}\varepsilon$ . Then

$$|\mathbf{x}_j - \mathbf{p}| \le |\mathbf{x}_j - \mathbf{x}_{k_m}| + |\mathbf{x}_{k_m} - \mathbf{p}| < \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon$$

whenever  $j \geq N$ . It follows that  $\mathbf{x}_j \to \mathbf{p}$  as  $j \to +\infty$ , as required.