

Module MAU23203: Analysis in Several Real
Variables

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Section 5: Continuous Functions of Several
Real Variables

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Contents

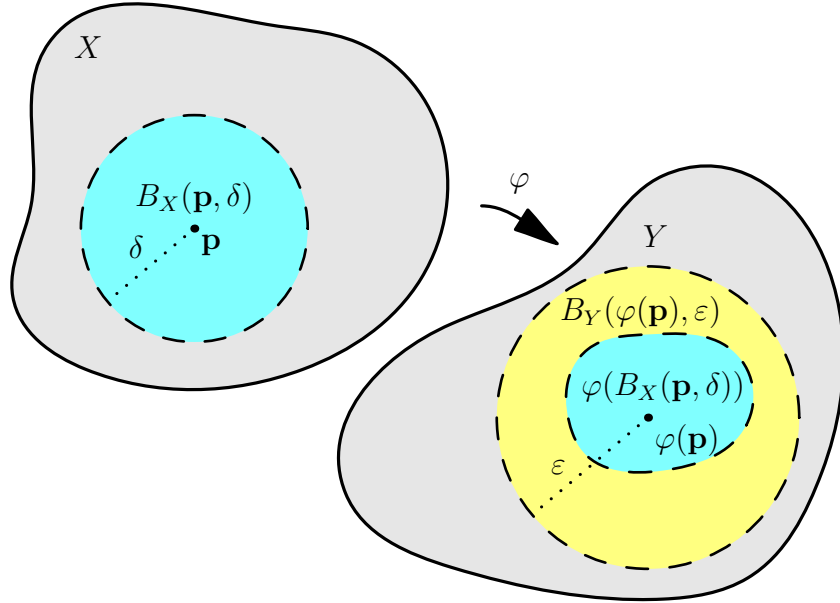
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5 Continuous Functions of Several Real Variables

5.1 The Concept and Basic Properties of Continuity

Definition Let X and Y be subsets of \mathbb{R}^m and \mathbb{R}^n respectively. A function $\varphi: X \rightarrow Y$ from X to Y is said to be *continuous* at a point \mathbf{p} of X if and only if, given any strictly positive real number ε , there exists some strictly positive real number δ such that $|\varphi(\mathbf{x}) - \varphi(\mathbf{p})| < \varepsilon$ whenever $\mathbf{x} \in X$ satisfies $|\mathbf{x} - \mathbf{p}| < \delta$.

The function $\varphi: X \rightarrow Y$ is said to be continuous on X if and only if it is continuous at every point \mathbf{p} of X .



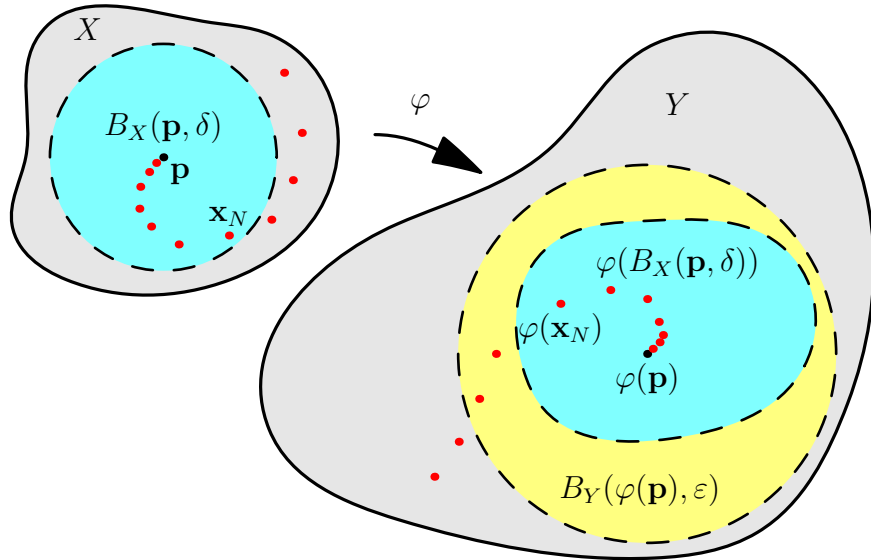
Proposition 5.1 Let X, Y and Z be subsets of Euclidean spaces, let $\varphi: X \rightarrow Y$ be a function from X to Y and let $\psi: Y \rightarrow Z$ be a function from Y to Z . Suppose that φ is continuous at some point \mathbf{p} of X and that ψ is continuous at $\varphi(\mathbf{p})$. Then the composition function $\psi \circ \varphi: X \rightarrow Z$ is continuous at \mathbf{p} .

Proof Let $\mathbf{q} = \varphi(\mathbf{p})$, and let some positive real number ε be given. Then there exists some positive real number η such that $|\psi(\mathbf{y}) - \psi(\mathbf{q})| < \varepsilon$ for all $\mathbf{y} \in Y$ satisfying $|\mathbf{y} - \mathbf{q}| < \eta$. But then there exists some positive real number δ such that $|\varphi(\mathbf{x}) - \mathbf{q}| < \eta$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$. It

follows that $|\psi(\varphi(\mathbf{x})) - \psi(\varphi(\mathbf{p}))| < \varepsilon$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$, and thus $\psi \circ \varphi$ is continuous at \mathbf{p} , as required. ■

Proposition 5.2 *Let X and Y be subsets of Euclidean spaces, and let $\varphi: X \rightarrow Y$ be a continuous function from X to Y . Let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \dots$ be an infinite sequence of points of X which converges to some point \mathbf{p} of X . Then the sequence $\varphi(\mathbf{x}_1), \varphi(\mathbf{x}_2), \varphi(\mathbf{x}_3), \dots$ converges to $\varphi(\mathbf{p})$.*

Proof Let some positive real number ε be given. The function φ is continuous at \mathbf{p} , and therefore there exists some positive real number δ such that $|\varphi(\mathbf{x}) - \varphi(\mathbf{p})| < \varepsilon$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$. Also the infinite se-



quence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \dots$ converges to the point \mathbf{p} , and therefore there exists some positive integer N such that $|\mathbf{x}_j - \mathbf{p}| < \delta$ whenever $j \geq N$. It follows that if $j \geq N$ then $|\varphi(\mathbf{x}_j) - \varphi(\mathbf{p})| < \varepsilon$. Thus the sequence $\varphi(\mathbf{x}_1), \varphi(\mathbf{x}_2), \varphi(\mathbf{x}_3), \dots$ converges to $\varphi(\mathbf{p})$, as required. ■

Let X and Y be subsets of \mathbb{R}^m and \mathbb{R}^n respectively, and let $\varphi: X \rightarrow Y$ be a function from X to Y . Then

$$\varphi(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}), \dots, f_n(\mathbf{x}))$$

for all $\mathbf{x} \in X$, where f_1, f_2, \dots, f_n are functions from X to \mathbb{R} , referred to as the *components* of the function φ .

Proposition 5.3 *Let X and Y be subsets of Euclidean spaces, and let $\mathbf{p} \in X$. A function $\varphi: X \rightarrow Y$ is continuous at the point \mathbf{p} if and only if its components are all continuous at \mathbf{p} .*

Proof Let Y be a subset of n -dimensional Euclidean space \mathbb{R}^n . Note that the i th component f_i of φ is given by $f_i = \pi_i \circ \varphi$, where $\pi_i: \mathbb{R}^n \rightarrow \mathbb{R}$ is the continuous function which maps $(y_1, y_2, \dots, y_n) \in \mathbb{R}^n$ onto its i th component y_i . Now any composition of continuous functions is continuous, by Proposition 5.1. Thus if φ is continuous at \mathbf{p} , then so are the components of φ .

Conversely suppose that the components of φ are continuous at $\mathbf{p} \in X$. Let some positive real number ε be given. Then there exist positive real numbers $\delta_1, \delta_2, \dots, \delta_n$ such that $|f_i(\mathbf{x}) - f_i(\mathbf{p})| < \varepsilon/\sqrt{n}$ for $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta_i$. Let δ be the minimum of $\delta_1, \delta_2, \dots, \delta_n$. If $\mathbf{x} \in X$ satisfies $|\mathbf{x} - \mathbf{p}| < \delta$ then

$$|\varphi(\mathbf{x}) - \varphi(\mathbf{p})|^2 = \sum_{i=1}^n |f_i(\mathbf{x}) - f_i(\mathbf{p})|^2 < \varepsilon^2,$$

and hence $|\varphi(\mathbf{x}) - \varphi(\mathbf{p})| < \varepsilon$. Thus the function φ is continuous at \mathbf{p} , as required. ■

Lemma 5.4 *Let functions $s: \mathbb{R}^2 \rightarrow \mathbb{R}$ and $m: \mathbb{R}^2 \rightarrow \mathbb{R}$ be defined so that $s(x, y) = x + y$ and $m(x, y) = xy$ for all real numbers x and y . Then the functions s and m are continuous.*

Proof Let $(u, v) \in \mathbb{R}^2$. We first show that $s: \mathbb{R}^2 \rightarrow \mathbb{R}$ is continuous at (u, v) . Let some positive real number ε be given. Let $\delta = \frac{1}{2}\varepsilon$. If (x, y) is any point of \mathbb{R}^2 whose distance from (u, v) is less than δ then $|x - u| < \delta$ and $|y - v| < \delta$, and hence

$$|s(x, y) - s(u, v)| = |x + y - u - v| \leq |x - u| + |y - v| < 2\delta = \varepsilon.$$

This shows that $s: \mathbb{R}^2 \rightarrow \mathbb{R}$ is continuous at (u, v) .

Next we show that $m: \mathbb{R}^2 \rightarrow \mathbb{R}$ is continuous at (u, v) . Let some positive real number ε be given. Now

$$m(x, y) - m(u, v) = xy - uv = (x - u)(y - v) + u(y - v) + (x - u)v.$$

for all points (x, y) of \mathbb{R}^2 . Thus if the distance from (x, y) to (u, v) is less than δ then $|x - u| < \delta$ and $|y - v| < \delta$, and hence $|m(x, y) - m(u, v)| < \delta^2 + (|u| + |v|)\delta$. Consequently if the positive real number δ is chosen to be the minimum of 1 and $\varepsilon/(1 + |u| + |v|)$ then $\delta^2 + (|u| + |v|)\delta \leq (1 + |u| + |v|)\delta \leq \varepsilon$, and thus $|m(x, y) - m(u, v)| < \varepsilon$ for all points (x, y) of \mathbb{R}^2 whose distance from (u, v) is less than δ . This shows that $m: \mathbb{R}^2 \rightarrow \mathbb{R}$ is continuous at (u, v) . ■

Proposition 5.5 *Let X be a subset of \mathbb{R}^n , and let $f: X \rightarrow \mathbb{R}$ and $g: X \rightarrow \mathbb{R}$ be continuous functions from X to \mathbb{R} . Then the functions $f + g$, $f - g$ and $f \cdot g$ are continuous. If in addition $g(\mathbf{x}) \neq 0$ for all $\mathbf{x} \in X$ then the quotient function f/g is continuous.*

Proof Note that $f + g = s \circ \psi$ and $f \cdot g = m \circ \psi$, where the functions $\psi: X \rightarrow \mathbb{R}^2$, $s: \mathbb{R}^2 \rightarrow \mathbb{R}$ and $m: \mathbb{R}^2 \rightarrow \mathbb{R}$ are defined so that $\psi(\mathbf{x}) = (f(\mathbf{x}), g(\mathbf{x}))$, $s(u, v) = u + v$ and $m(u, v) = uv$ for all $\mathbf{x} \in X$ and $u, v \in \mathbb{R}$. It follows from Proposition 5.3, Lemma 5.4 and Proposition 5.1 that $f + g$ and $f \cdot g$ are continuous, being compositions of continuous functions. Now $f - g = f + (-g)$, and both f and $-g$ are continuous. Therefore $f - g$ is continuous.

Now suppose that $g(\mathbf{x}) \neq 0$ for all $\mathbf{x} \in X$. Note that $1/g = r \circ g$, where $r: \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}$ is the reciprocal function, defined so that $r(t) = 1/t$ for all non-zero real numbers t . Now the reciprocal function r is continuous. Thus the function $1/g$ is a composition of continuous functions and is thus continuous. But then, using the fact that a product of continuous real-valued functions is continuous, we deduce that f/g is continuous. ■

Example Consider the function $\varphi: \mathbb{R}^2 \setminus \{(0, 0)\} \rightarrow \mathbb{R}^2$ defined so that

$$\varphi(x, y) = \left(\frac{x}{x^2 + y^2}, \frac{-y}{x^2 + y^2} \right)$$

for all real numbers x and y that are not both zero. The continuity of the components of this function φ follows from straightforward applications of Proposition 5.5. It then follows from Proposition 5.3 that the function φ is continuous on $\mathbb{R}^2 \setminus \{(0, 0)\}$.

Lemma 5.6 *Let X be a subset of \mathbb{R}^m , let $\varphi: X \rightarrow \mathbb{R}^n$ be a continuous function mapping X into \mathbb{R}^n , and let $|\varphi|: X \rightarrow \mathbb{R}$ be the real-valued function on X defined such that $|\varphi|(\mathbf{x}) = |\varphi(\mathbf{x})|$ for all $\mathbf{x} \in X$. Then the real-valued function $|\varphi|$ is continuous on X .*

Proof Let \mathbf{x} and \mathbf{p} be points of X . Then

$$|\varphi(\mathbf{x})| = |(\varphi(\mathbf{x}) - \varphi(\mathbf{p})) + \varphi(\mathbf{p})| \leq |\varphi(\mathbf{x}) - \varphi(\mathbf{p})| + |\varphi(\mathbf{p})|$$

and

$$|\varphi(\mathbf{p})| = |(\varphi(\mathbf{p}) - \varphi(\mathbf{x})) + \varphi(\mathbf{x})| \leq |\varphi(\mathbf{x}) - \varphi(\mathbf{p})| + |\varphi(\mathbf{x})|,$$

and therefore

$$\left| |\varphi(\mathbf{x})| - |\varphi(\mathbf{p})| \right| \leq |\varphi(\mathbf{x}) - \varphi(\mathbf{p})|.$$

The result now follows on applying the definition of continuity, using the above inequality. Indeed let \mathbf{p} be a point of X , and let some positive real number ε be given. Then there exists a positive real number δ small enough to ensure that $|\varphi(\mathbf{x}) - \varphi(\mathbf{p})| < \varepsilon$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$. But then

$$\left| |\varphi(\mathbf{x})| - |\varphi(\mathbf{p})| \right| \leq |\varphi(\mathbf{x}) - \varphi(\mathbf{p})| < \varepsilon$$

for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$, and thus the function $|\varphi|$ is continuous, as required. ■

5.2 Continuous Functions and Open Sets

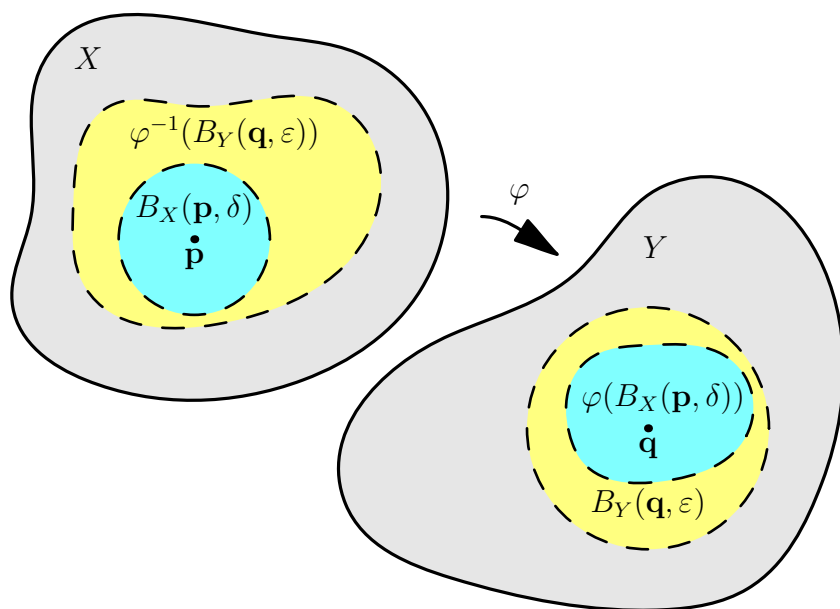
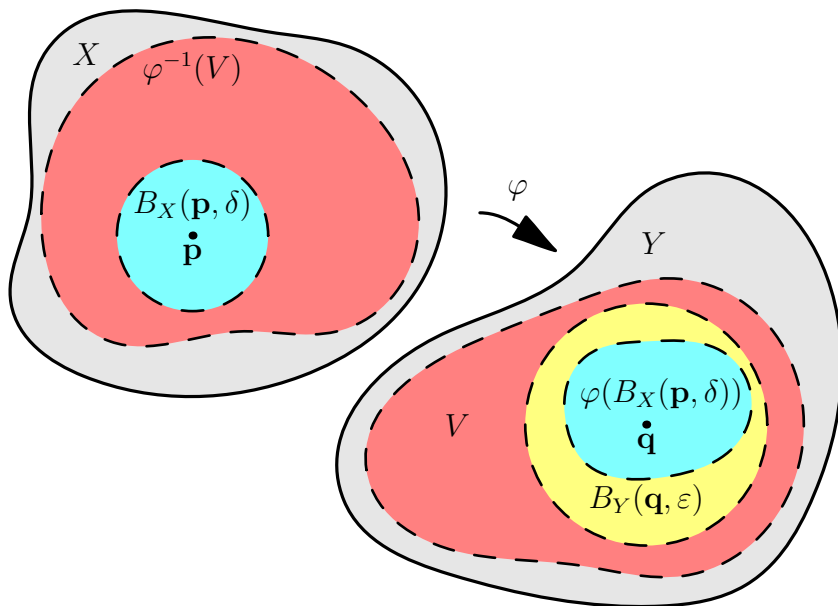
Let X and Y be subsets of \mathbb{R}^m and \mathbb{R}^n , and let $\varphi: X \rightarrow Y$ be a function from X to Y . We recall that the function φ is continuous at a point \mathbf{p} of X if and only if, given any positive real number ε , there exists some positive real number δ such that $|\varphi(\mathbf{x}) - \varphi(\mathbf{p})| < \varepsilon$ for all points \mathbf{x} of X satisfying $|\mathbf{x} - \mathbf{p}| < \delta$. Thus the function $\varphi: X \rightarrow Y$ is continuous at \mathbf{p} if and only if, given any positive real number ε , there exists some positive real number δ such that the function φ maps the open ball $B_X(\mathbf{p}, \delta)$ in X of radius δ centred on the point \mathbf{p} into the open ball $B_Y(\mathbf{q}, \varepsilon)$ in Y of radius ε centered on the point \mathbf{q} , where $\mathbf{q} = \varphi(\mathbf{p})$.

Given any function $\varphi: X \rightarrow Y$, we denote by $\varphi^{-1}(V)$ the *preimage* of a subset V of Y under the map φ , defined so that $\varphi^{-1}(V) = \{\mathbf{x} \in X : \varphi(\mathbf{x}) \in V\}$.

Proposition 5.7 *Let X and Y be subsets of \mathbb{R}^m and \mathbb{R}^n , and let $\varphi: X \rightarrow Y$ be a function from X to Y . The function φ is continuous if and only if $\varphi^{-1}(V)$ is open in X for every open subset V of Y .*

Proof Suppose that $\varphi: X \rightarrow Y$ is continuous. Let V be an open set in Y . We must show that $\varphi^{-1}(V)$ is open in X . Let \mathbf{p} be a point of $\varphi^{-1}(V)$, and let $\mathbf{q} = \varphi(\mathbf{p})$. Then $\mathbf{q} \in V$. But V is open, hence there exists some positive real number ε with the property that $B_Y(\mathbf{q}, \varepsilon) \subset V$. But φ is continuous at \mathbf{p} . Therefore there exists some positive real number δ such that φ maps $B_X(\mathbf{p}, \delta)$ into $B_Y(\mathbf{q}, \varepsilon)$. Thus $\varphi(\mathbf{x}) \in V$ for all $\mathbf{x} \in B_X(\mathbf{p}, \delta)$, showing that $B_X(\mathbf{p}, \delta) \subset \varphi^{-1}(V)$. This shows that $\varphi^{-1}(V)$ is open in X for every open set V in Y .

Conversely suppose that $\varphi: X \rightarrow Y$ is a function with the property that $\varphi^{-1}(V)$ is open in X for every open set V in Y . Let $\mathbf{p} \in X$, and let $\mathbf{q} = \varphi(\mathbf{p})$. We must show that φ is continuous at \mathbf{p} . Let some positive real number ε be given. Then $B_Y(\mathbf{q}, \varepsilon)$ is an open set in Y , by Lemma 4.1, hence $\varphi^{-1}(B_Y(\mathbf{q}, \varepsilon))$ is an open set in X which contains \mathbf{p} . It follows that there exists some positive



real number δ such that $B_X(\mathbf{p}, \delta) \subset \varphi^{-1}(B_Y(\mathbf{q}, \varepsilon))$. Thus, given any positive real number ε , there exists some positive real number δ such that φ maps $B_X(\mathbf{p}, \delta)$ into $B_Y(\mathbf{q}, \varepsilon)$. We conclude that φ is continuous at the point \mathbf{p} , as required. ■

Let X be a subset of \mathbb{R}^n , let $f: X \rightarrow \mathbb{R}$ be continuous, and let c be some real number. Then the sets

$$\{\mathbf{x} \in X : f(\mathbf{x}) > c\}$$

and

$$\{\mathbf{x} \in X : f(\mathbf{x}) < c\}$$

are open in X , and, given real numbers a and b satisfying $a < b$, the set

$$\{\mathbf{x} \in X : a < f(\mathbf{x}) < b\}$$

is open in X .

Again let X be a subset of \mathbb{R}^n , let $f: X \rightarrow \mathbb{R}$ be continuous, and let c be some real number. Now a subset of X is closed in X if and only if its complement is open in X . Consequently the sets

$$\{\mathbf{x} \in X : f(\mathbf{x}) \leq c\}$$

and

$$\{\mathbf{x} \in X : f(\mathbf{x}) \geq c\},$$

being the complements in X of sets that are open in X , must themselves be closed in X . It follows that that set

$$\{\mathbf{x} \in X : f(\mathbf{x}) = c\},$$

being the intersection of two subsets X that are closed in X , must itself be closed in X .

5.3 The Multidimensional Extreme Value Theorem

Lemma 5.8 *Let X be a non-empty closed bounded set in \mathbb{R}^m , and let $f: X \rightarrow \mathbb{R}$ be a continuous real-valued function defined on X . Suppose that the set of values of the function f on X is bounded below. Then there exists a point \mathbf{u} of X such that $f(\mathbf{u}) \leq f(\mathbf{x})$ for all $\mathbf{x} \in X$.*

Proof Let

$$L = \inf\{f(\mathbf{x}) : \mathbf{x} \in X\}.$$

Then there exists an infinite sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \dots$ in X such that

$$f(\mathbf{x}_j) < L + \frac{1}{j}$$

for all positive integers j . It follows from the multidimensional Bolzano-Weierstrass Theorem (Theorem 3.5) that this sequence has a subsequence $\mathbf{x}_{k_1}, \mathbf{x}_{k_2}, \mathbf{x}_{k_3}, \dots$ which converges to some point \mathbf{u} of \mathbb{R}^m .

Now the point \mathbf{u} belongs to X because X is closed (see Lemma 4.7). Also

$$L \leq f(\mathbf{x}_{k_j}) < L + \frac{1}{k_j}$$

for all positive integers j . It follows that $\lim_{j \rightarrow +\infty} f(\mathbf{x}_{k_j}) = L$. Consequently

$$f(\mathbf{u}) = f\left(\lim_{j \rightarrow +\infty} \mathbf{x}_{k_j}\right) = \lim_{j \rightarrow +\infty} f(\mathbf{x}_{k_j}) = L$$

(see Proposition 5.2). It follows therefore that $f(\mathbf{x}) \geq f(\mathbf{u})$ for all $\mathbf{x} \in X$. Thus the function f attains a minimum value at the point \mathbf{u} of X , which is what we were required to prove. ■

Lemma 5.9 *Let X be a non-empty closed bounded set in \mathbb{R}^m , and let $\varphi: X \rightarrow \mathbb{R}^n$ be a continuous function mapping X into \mathbb{R}^n . Then there exists a positive real number M with the property that $|\varphi(\mathbf{x})| \leq M$ for all $\mathbf{x} \in X$.*

Proof Let $g: X \rightarrow \mathbb{R}$ be defined such that

$$g(\mathbf{x}) = \frac{1}{1 + |\varphi(\mathbf{x})|}$$

for all $\mathbf{x} \in X$. Now the real-valued function mapping each $\mathbf{x} \in X$ to $|\varphi(\mathbf{x})|$ is continuous (see Lemma 5.6) and quotients of continuous real-valued functions are continuous where they are defined (see Lemma 5.5). It follows that the function $g: X \rightarrow \mathbb{R}$ is continuous. Moreover the values of this function are bounded below by zero. Consequently there exists some point \mathbf{w} of X with the property that $g(\mathbf{x}) \geq g(\mathbf{w})$ for all $\mathbf{x} \in X$ (see Lemma 5.8). Let $M = |\varphi(\mathbf{w})|$. Then $|\varphi(\mathbf{x})| \leq M$ for all $\mathbf{x} \in X$. The result follows. ■

Theorem 5.10 (The Multidimensional Extreme Value Theorem)

Let X be a non-empty closed bounded set in \mathbb{R}^m , and let $f: X \rightarrow \mathbb{R}$ be a continuous real-valued function defined on X . Then there exist points \mathbf{u} and \mathbf{v} of X such that $f(\mathbf{u}) \leq f(\mathbf{x}) \leq f(\mathbf{v})$ for all $\mathbf{x} \in X$.

Proof It follows from Lemma 5.9 that there exists positive real number M with the property that $-M \leq f(\mathbf{x}) \leq M$ for all $\mathbf{x} \in X$. Thus the set of values of the function f is bounded above and below on X . Consequently there exist points \mathbf{u} and \mathbf{v} where the functions f and $-f$ respectively attain their minimum values on the set X (see Lemma 5.8). The result follows. ■

5.4 Uniform Continuity for Functions of Several Real Variables

Definition Let X be a subset of \mathbb{R}^m . A function $\varphi: X \rightarrow \mathbb{R}^n$ from X to \mathbb{R}^n is said to be *uniformly continuous* if, given any positive real number ε , there exists some positive real number δ (whose value does not depend on either \mathbf{y} or \mathbf{z}) such that $|\varphi(\mathbf{y}) - \varphi(\mathbf{z})| < \varepsilon$ for all points \mathbf{y} and \mathbf{z} of X satisfying $|\mathbf{y} - \mathbf{z}| < \delta$.

Theorem 5.11 *Let X be a non-empty closed bounded set in \mathbb{R}^m . Then any continuous function $\varphi: X \rightarrow \mathbb{R}^n$ is uniformly continuous.*

Proof Let some positive real number ε be given. Suppose that there did not exist any positive real number δ small enough to ensure that $|\varphi(\mathbf{y}) - \varphi(\mathbf{z})| < \varepsilon$ for all points \mathbf{y} and \mathbf{z} of the set X satisfying $|\mathbf{y} - \mathbf{z}| < \delta$. Then, for each positive integer j , there would exist points \mathbf{u}_j and \mathbf{v}_j in X such that $|\mathbf{u}_j - \mathbf{v}_j| < 1/j$ and $|\varphi(\mathbf{u}_j) - \varphi(\mathbf{v}_j)| \geq \varepsilon$. But the sequence $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots$ would be bounded, since X is bounded, and thus would possess a subsequence $\mathbf{u}_{k_1}, \mathbf{u}_{k_2}, \mathbf{u}_{k_3}, \dots$ converging to some point \mathbf{p} (Theorem 3.5). Moreover $\mathbf{p} \in X$, because X is closed in \mathbb{R}^n . The sequence $\mathbf{v}_{k_1}, \mathbf{v}_{k_2}, \mathbf{v}_{k_3}, \dots$ would also converge to \mathbf{p} , because

$$\lim_{j \rightarrow +\infty} |\mathbf{v}_{k_j} - \mathbf{u}_{k_j}| = 0.$$

But then the sequences

$$\varphi(\mathbf{u}_{k_1}), \varphi(\mathbf{u}_{k_2}), \varphi(\mathbf{u}_{k_3}), \dots$$

and

$$\varphi(\mathbf{v}_{k_1}), \varphi(\mathbf{v}_{k_2}), \varphi(\mathbf{v}_{k_3}), \dots$$

would both converge to $\varphi(\mathbf{p})$, because φ is continuous (see Proposition 5.2). Therefore

$$\lim_{j \rightarrow +\infty} |\varphi(\mathbf{u}_{k_j}) - \varphi(\mathbf{v}_{k_j})| = 0.$$

But, assuming that no positive real number δ could be found satisfying the stated requirements, the points \mathbf{u}_j and \mathbf{v}_j had been chosen for all positive

integers j so that $|\mathbf{u}_j - \mathbf{v}_j| < 1/j$ and $|\varphi(\mathbf{u}_j) - \varphi(\mathbf{v}_j)| \geq \varepsilon$. Consequently $\varphi(\mathbf{u}_{k_j})$ and $\varphi(\mathbf{v}_{k_j})$ could not both converge to $\varphi(\mathbf{p})$ as j increases to infinity. Thus the assumption that no positive real number δ would have the required property would lead to a contradiction. We conclude therefore that, in order to avoid arriving at this contradiction, there must exist some positive real number δ such that $|\varphi(\mathbf{y}) - \varphi(\mathbf{z})| < \varepsilon$ for all points \mathbf{y} and \mathbf{z} of the set X satisfying $|\mathbf{y} - \mathbf{z}| < \delta$, as required. ■