MAU23203—Analysis in Several Variables School of Mathematics, Trinity College Michaelmas Term 2023 Additional Presentation: Material Related to the Bolzano-Weierstrass Theorem

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An Alternative Proof of the Multidimensional Bolzano-Weierstrass Theorem

Given a point $\mathbf{p}$ in $\mathbb{R}^n$, and given a positive real number s, let the n-dimensional hypercube $H(\mathbf{p}, s)$ be defined so that

$$\begin{aligned} H(\mathbf{p}, s) &= \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : \\ &\qquad (\mathbf{p})_i \leq x_i \leq (\mathbf{p})_i + s \text{ for } i = 1, 2, \ldots, n \}. \end{aligned}$$

(Here, and in what follows, $(\mathbf{p})_i$ denotes the ith Cartesian coordinate of the point (or *position vector*) $\mathbf{p}$. This hypercube has sides of length s. In cases where $n = 2$ it is square with sides parallel to the coordinate axes; in cases where $n = 3$ it is a cube with sides parallel to the coordinate axes.

Let $H(\mathbf{p}, s)$ be a hypercube with sides of length s determined by a corner $\mathbf{p}$ (where Cartesian coordinates are minimized on the hypercube). This hypercube can be dissected into 2^n hypercubes with sides of length $\frac{1}{2}s$, where each of these hypercubes can be represented in the form $H(\mathbf{q}, \frac{1}{2}s)$ for some point $\mathbf{q}$ whose ith coordinate $(\mathbf{q})_i$ is equal to one or other of the numbers $(\mathbf{p})_i$ and $(\mathbf{p})_i + \frac{1}{2}s$, where $(\mathbf{p})_i$ here denotes the ith component of the determining corner $\mathbf{p}$ of the hypercube being dissected. Let us refer to the hypercubes formed in this fashion as the hypercubes resulting from the *natural dissection* of the hypercube $H(\mathbf{p}, s)$ into hypercubes with sides of length $\frac{1}{2}s$.

Let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ be an infinite sequence of points in $\mathbb{R}^n$. We say that a hypercube $H(\mathbf{p}, s)$ contains *infinitely many* members of this sequence if there are infinitely many positive integers j for which $\mathbf{x}_j \in H(\mathbf{p}, s)$.

Now if a hypercube $H(\mathbf{p}, s)$ with sides of length s contains infinitely many members of the infinite sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$, then the same must be true of at least one of the 2^n hypercubes with sides of length $\frac{1}{2}s$ arising from the natural dissection of the hypercube $H(\mathbf{p}, s)$. It follows from this that if a hypercube $H(\mathbf{p}, s)$ with sides of length s contains all members of a bounded infinite sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ of points in $\mathbb{R}^n$, then it must be possible to construct an infinite sequence $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \ldots$ of points of $\mathbb{R}^n$, where $H(\mathbf{p}_1, s/2)$ arises from the natural dissection of $H(\mathbf{p}, s)$, the hypercube $H(\mathbf{p}_k, s/2^k)$ arises from the natural dissection of the hypercube $H(\mathbf{p}_{k-1}, s/2^{k-1})$ for each positive integer k greater than 1, and furthermore each hypercube $H(\mathbf{p}_k, s/2^k)$ is determined so as to contain infinitely many members of the infinite sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$.

Now $H(\mathbf{p}_{k-1}, s/2^{k-1}) \subset H(\mathbf{p}_k, s/2^k)$ for each positive integer k greater than 1, and $H(\mathbf{p}_k, s/2^k) \subset H(\mathbf{p}, s)$ for all positive integers k. Also, for each positive integer i between 1 and n,

$$(\mathbf{p}_1)_i \leq (\mathbf{p}_2)_i \leq (\mathbf{p}_3)_i \leq \cdots$$

(where $(\mathbf{p}_k)_i$ denotes the ith Cartesian component of the point $\mathbf{p}_k$ for $i = 1, 2, \ldots, n$ and for all positive integers k). Furthermore (where $(\mathbf{p}_k)_i \leq (\mathbf{p})_i + s$ for all positive integers k. Now any non-decreasing sequence of real numbers that is bounded above must converge to some real number. Accordingly there must exist a point $\mathbf{q}$ of $\mathbb{R}^n$ characterized by the property that

$$\lim_{k \to +\infty} (\mathbf{p}_k)_i = (\mathbf{q})_i \quad \text{for } i = 1, 2, \ldots, k.$$

We now show that, given any positive real number ε, there must exist infinitely many positive integers j for which $|\mathbf{x}_j - \mathbf{q}| < \varepsilon$. Now the convergence of $(\mathbf{p}_k)_i$ to $(\mathbf{q})_i$ for $i = 1, 2, \ldots, n$ as $k \to +\infty$ ensures the existence of some positive integer N with the property that

$$(\mathbf{q})_i - \frac{\varepsilon}{\sqrt{n}} < (\mathbf{p}_k)_i \leq (\mathbf{q})_i$$

for $i = 1, 2, \ldots, k$ and for all positive integers k satisfying $k \geq N$. (Note that $(\mathbf{p}_k)_i \leq (\mathbf{q})_i$ for $i = 1, 2, \ldots, n$ and for all positive integers k, because the infinite sequence consisting of the ith components of the points $\mathbf{p}_k$ is non-decreasing for $i = 1, 2, \ldots, n$.)

Now the positive integer N may be chosen large enough to ensure that $s/2^N < \varepsilon/\sqrt{n}$. It follows that if $\mathbf{u}$ is a point of $H(\mathbf{p}_k, s/2^k)$, where $k \geq N$, then

$$(\mathbf{q})_i - \frac{\varepsilon}{\sqrt{n}} < (\mathbf{u})_i < (\mathbf{q})_i - \frac{\varepsilon}{\sqrt{n}}$$

for $i = 1, 2, \ldots, n$. Consequently

$$|\mathbf{u} - \mathbf{q}|^2 = \sum_{i=1}^{n} \Big((\mathbf{u})_i - (\mathbf{q})_i \Big)^2 \leq \varepsilon^2,$$

and therefore $|\mathbf{u} - \mathbf{q}| < \varepsilon$. But, for each positive integer k, infinitely many members of the infinite sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ belong to the hypercube $H(\mathbf{p}_k, 2^k)$. Consequently $|\mathbf{x}_j - \mathbf{q}| < \varepsilon$ for infinitely many positive integers j.

We have now shown that, given any positive real number ε, there exist infinitely many positive integers j for which $|\mathbf{x}_j - \mathbf{q}| < \varepsilon$. It follows that there must exist an increasing sequence

$$j_1, j_2, j_3, \ldots$$

of positive integers which is such as to ensure that $|\mathbf{x}_{j_m} - \mathbf{q}| < 1/2^m$ for all positive integers j. Then the subsequence $\mathbf{x}_{m_1}, \mathbf{x}_{m_2}, \mathbf{x}_{m_3} \ldots$ of the given bounded sequence converges to the point $\mathbf{q}$. We have therefore completed a proof of the multidimensional Bolzano-Weierstrass Theorem.