MAU23203—Analysis in Several Variables
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Section 1: The Real Number System

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1. The Real Number System

1.1. Basic Properties of the Real Number System

The real numbers $\mathbb{R}$ constitute a *field* with respect to the usual operations of addition and multiplication. In other words, the real number system satisfies all of the following properties: the operations of addition and multiplication satisfy the usual commutative, associative and distributive laws, so that $x + y = y + x$, $xy = yx$, $(x + y) + z = x + (y + z)$, $(xy)z = x(yz)$ and $(x + y)z = xz + yz$ for all real numbers x, y and z; there exist real numbers 0 and 1 characterized by the properties that $0 + x = x$ and $1x = x$ for all real numbers x; given any real number x, there exists a real number $-x$ characterized by the property that $x + (-x) = 0$; given any non-zero real number x there exists a real number x^{-1} characterized by the property that $xx^{-1} = 1$.

In the field $\mathbb{R}$ of real numbers operations of subtraction of real numbers, and division of real numbers by non-zero real numbers are defined so that $x - y = x + (-y)$ and $x/z = xz^{-1}$ for all real numbers x and y and for all non-zero real numbers z. A variety of other algebraic identities and properties follow as consequences of those just stated: analogous identities and properties are valid in any field.

The real numbers $\mathbb{R}$ constitute an *ordered field* with respect to the usual operations of addition and multiplication and the usual ordering. This statement amounts to asserting that, in addition to having operations of addition and multiplication satisfying the properties already described, there is an ordering $<$ on the real numbers which satisfies all the following properties:—

- given two real numbers x and y, exactly one of the ordering relations $x < y$, $x = y$, $y < x$ must hold for x and y (*Trichotomy Law*);
- if real numbers x, y and z satisfy both $x < y$ and $y < z$ then $x < z$ (*Transitivity Law*);
- if x and y are real numbers that satisfy $x < y$ then $x + z < y + z$ for all real numbers z;
- if x and y are real numbers satisfying $x > 0$ and $y > 0$ then $xy > 0$.

Let S be a subset of the set $\mathbb{R}$ of real numbers. A real number u is said to be an *upper bound* of the set S if $x \leq u$ for all $x \in S$. The set S is said to be *bounded above* if such an upper bound exists. Similarly a real number l is said to be a *lower bound* of the set S if $x \geq l$ for all $x \in S$. The set S is said to be *bounded below* if such a lower bound exists.

Definition

Let S be a subset of the set $\mathbb{R}$ of real numbers that is bounded above. A real number s is said to be the *least upper bound* (or *supremum*) of S (and is denoted by $\sup S$) if s is an upper bound of S and $s \leq u$ for all upper bounds u of S.

Definition

Let S be a subset of the set $\mathbb{R}$ of real numbers that is bounded below. A real number t is said to be the *greatest lower bound* (or *infimum*) of S (and is denoted by $\inf S$) if t is a lower bound of S and $t \geq l$ for all lower bounds l of S.

Least Upper Bound Principle

Given any non-empty set S of real numbers that is bounded above, there exists a real number $\sup S$ that is the least upper bound for the set S.

It follows as a consequence of the Least Upper Bound Principle that, given any non-empty set S of real numbers that is bounded below, there exists a real number $\inf S$ that is the greatest lower bound for the set S. Indeed, given any non-empty set S of real numbers that is bounded below, let $T = \{-x : x \in S\}$. Then the set T is non-empty and bounded above, and therefore there exists a least upper bound $\sup T$ for the set T. It is then a straightforward exercise to verify that $\inf S = -\sup T$.

In Dedekind's construction, each irrational number is represented as a decomposition of the collection of rational numbers into two classes (or sets) L and R, where each rational number belongs to exactly one of the two classes L and R, and where each rational number belonging to L is less than all the rational numbers belonging to R. Each such decomposition of the collection of rational numbers is referred to as a *Dedekind section*.

In Cantor's construction, expressed in more contemporary language, each real number is constructed as an equivalence class of Cauchy sequences of rational numbers. An infinite sequence $q_1, q_2, q_3, \ldots$ of rational numbers is a Cauchy sequence if, given any positive integer m, there exists some positive integer N such that $|q_j - q_k| < 1/m$ whenever $j \geq N$ and $k \geq N$. Two such Cauchy sequences of rational numbers $q_1, q_2, q_3, \ldots$ and $r_1, r_2, r_3, \ldots$ are said to be equivalent if, given any positive integer m, there exists some positive integer N such that $|q_j - r_j| < 1/m$ whenever $j \geq N$. (Note that, in order to avoid circularity, in phrasing these definitions, it is necessary to use definitions where quantities are made less than the reciprocal $1/m$ of some positive integer m in place of the "positive real number ε".) One can show that the definition of equivalence of Cauchy sequences previously stated is an equivalence relation. The resulting equivalence classes of Cauchy sequences are identified with real numbers in Cantor's construction of the real number system.