

**MAU23203—Analysis in Several Variables**  
**School of Mathematics, Trinity College**  
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**Section 6: Limits of Functions of Several**  
**Real Variables**

Trinity College Dublin

### 6. Limits of Functions of Several Real Variables

#### 6.1. Limit Points of Subsets of Euclidean Spaces

##### Definition

Let  $X$  be a subset of  $n$ -dimensional Euclidean space  $\mathbb{R}^n$ , and let  $\mathbf{p} \in \mathbb{R}^n$ . The point  $\mathbf{p}$  is said to be a *limit point* of the set  $X$  if, given any positive real number  $\delta$ , there exists some point  $\mathbf{x}$  of  $X$  for which  $0 < |\mathbf{x} - \mathbf{p}| < \delta$ .

## 6.2. Basic Properties of Limits of Functions of Several Real Variables

**Definition**

Let  $X$  be a subset of  $m$ -dimensional Euclidean space  $\mathbb{R}^m$ , let  $\varphi: X \rightarrow \mathbb{R}^n$  be a function mapping the set  $X$  into  $n$ -dimensional Euclidean space  $\mathbb{R}^n$ , let  $\mathbf{p}$  be a limit point of the set  $X$ , and let  $\mathbf{v}$  be a vector in  $\mathbb{R}^n$ . The point  $\mathbf{v}$  is said to be the *limit* of  $\varphi(\mathbf{x})$ , as  $\mathbf{x}$  tends to  $\mathbf{p}$  in  $X$ , if and only if, given any strictly positive real number  $\varepsilon$ , there exists some strictly positive real number  $\delta$  such that  $|\varphi(\mathbf{x}) - \mathbf{v}| < \varepsilon$  whenever  $\mathbf{x} \in X$  satisfies  $0 < |\mathbf{x} - \mathbf{p}| < \delta$ .

## 6. Limits of Functions of Several Real Variables (continued)

Let  $X$  be a subset of  $m$ -dimensional Euclidean space  $\mathbb{R}^m$ , let  $\varphi: X \rightarrow \mathbb{R}^n$  be a function mapping the set  $X$  into  $n$ -dimensional Euclidean space  $\mathbb{R}^n$ , let  $\mathbf{p}$  be a limit point of the set  $X$ , and let  $\mathbf{v}$  be a vector in  $\mathbb{R}^n$ . If  $\mathbf{v}$  is the limit of  $\varphi(\mathbf{x})$  as  $\mathbf{x}$  tends to  $\mathbf{p}$  in  $X$  then we can denote this fact by writing  $\lim_{\mathbf{x} \rightarrow \mathbf{p}} \varphi(\mathbf{x}) = \mathbf{v}$ .

**Proposition 6.1**

*Let  $X$  be a subset of  $\mathbb{R}^m$ , let  $\mathbf{p}$  be a limit point of  $X$ , and let  $\mathbf{v}$  be a vector in  $\mathbb{R}^n$ . A function  $\varphi: X \rightarrow \mathbb{R}^n$  has the property that*

$$\lim_{\mathbf{x} \rightarrow \mathbf{p}} \varphi(\mathbf{x}) = \mathbf{v}$$

*if and only if*

$$\lim_{\mathbf{x} \rightarrow \mathbf{p}} f_i(\mathbf{x}) = v_i$$

*for  $i = 1, 2, \dots, n$ , where  $f_1, f_2, \dots, f_n$  are the components of the function  $\varphi$  and  $\mathbf{v} = (v_1, v_2, \dots, v_n)$ .*

**Proof**

Suppose that  $\lim_{\mathbf{x} \rightarrow \mathbf{p}} \varphi(\mathbf{x}) = \mathbf{v}$ . Let  $i$  be an integer between 1 and  $n$ , and let some positive real number  $\varepsilon$  be given. Then there exists some positive real number  $\delta$  such that  $|\varphi(\mathbf{x}) - \mathbf{v}| < \varepsilon$  whenever  $0 < |\mathbf{x} - \mathbf{p}| < \delta$ . It then follows from the definition of the Euclidean norm that

$$|f_i(\mathbf{x}) - v_i| \leq |\varphi(\mathbf{x}) - \mathbf{v}| < \varepsilon$$

whenever  $0 < |\mathbf{x} - \mathbf{p}| < \delta$ . Thus if  $\lim_{\mathbf{x} \rightarrow \mathbf{p}} \varphi(\mathbf{x}) = \mathbf{v}$  then

$$\lim_{\mathbf{x} \rightarrow \mathbf{p}} f_i(\mathbf{x}) = v_i \text{ for } i = 1, 2, \dots, n.$$

## 6. Limits of Functions of Several Real Variables (continued)

Conversely suppose that

$$\lim_{\mathbf{x} \rightarrow \mathbf{p}} f_i(\mathbf{x}) = v_i$$

for  $i = 1, 2, \dots, n$ . Let some positive real number  $\varepsilon$  be given. Then there exist positive real numbers  $\delta_1, \delta_2, \dots, \delta_n$  such that

$|f_i(\mathbf{x}) - v_i| < \varepsilon/\sqrt{n}$  for  $\mathbf{x} \in X$  satisfying  $0 < |\mathbf{x} - \mathbf{p}| < \delta_i$ . Let  $\delta$  be the minimum of  $\delta_1, \delta_2, \dots, \delta_n$ . If  $\mathbf{x} \in X$  satisfies  $0 < |\mathbf{x} - \mathbf{p}| < \delta$  then

$$|\varphi(\mathbf{x}) - \mathbf{v}|^2 = \sum_{i=1}^n (f_i(\mathbf{x}) - v_i)^2 < \varepsilon^2,$$

and hence  $|\varphi(\mathbf{x}) - \mathbf{v}| < \varepsilon$ . Thus

$$\lim_{\mathbf{x} \rightarrow \mathbf{p}} \varphi(\mathbf{x}) = \mathbf{v},$$

as required. ■

**Proposition 6.2**

*Let  $X$  be a subset of  $m$ -dimensional Euclidean space  $\mathbb{R}^m$ , let  $\varphi: X \rightarrow \mathbb{R}^n$  and  $\psi: X \rightarrow \mathbb{R}^n$  be functions mapping  $X$  into  $n$ -dimensional Euclidean space  $\mathbb{R}^n$ , let  $\mathbf{p}$  be a limit point of  $X$ , and let  $\mathbf{v}$  and  $\mathbf{w}$  be points of  $\mathbb{R}^n$ . Suppose that*

$$\lim_{\mathbf{x} \rightarrow \mathbf{p}} \varphi(\mathbf{x}) = \mathbf{v}$$

*and*

$$\lim_{\mathbf{x} \rightarrow \mathbf{p}} \psi(\mathbf{x}) = \mathbf{w}.$$

*Then*

$$\lim_{\mathbf{x} \rightarrow \mathbf{p}} (\varphi(\mathbf{x}) + \psi(\mathbf{x})) = \mathbf{v} + \mathbf{w}.$$



**Proof**

Let some strictly positive real number  $\varepsilon$  be given. Then there exist strictly positive real numbers  $\delta_1$  and  $\delta_2$  such that

$$|\varphi(\mathbf{x}) - \mathbf{v}| < \frac{1}{2}\varepsilon$$

whenever  $\mathbf{x} \in X$  satisfies  $0 < |\mathbf{x} - \mathbf{p}| < \delta_1$  and

$$|\psi(\mathbf{x}) - \mathbf{w}| < \frac{1}{2}\varepsilon$$

whenever  $\mathbf{x} \in X$  satisfies  $0 < |\mathbf{x} - \mathbf{p}| < \delta_2$ .

## 6. Limits of Functions of Several Real Variables (continued)

Let  $\delta$  be the minimum of  $\delta_1$  and  $\delta_2$ . Then  $\delta > 0$ , and if  $\mathbf{x} \in X$  satisfies  $0 < |\mathbf{x} - \mathbf{p}| < \delta$  then

$$|\varphi(\mathbf{x}) - \mathbf{v}| < \frac{1}{2}\varepsilon$$

and

$$|\psi(\mathbf{x}) - \mathbf{w}| < \frac{1}{2}\varepsilon,$$

and therefore

$$\begin{aligned} |\varphi(\mathbf{x}) + \psi(\mathbf{x}) - (\mathbf{v} + \mathbf{w})| &\leq |\varphi(\mathbf{x}) - \mathbf{v}| + |\psi(\mathbf{x}) - \mathbf{w}| \\ &< \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon. \end{aligned}$$

It follows that

$$\lim_{\mathbf{x} \rightarrow \mathbf{p}} (\varphi(\mathbf{x}) + \psi(\mathbf{x})) = \mathbf{v} + \mathbf{w},$$

as required. ■

**Lemma 6.3**

*Let  $X$  and  $Y$  be subsets of  $\mathbb{R}^m$  and  $\mathbb{R}^n$  respectively, let  $\mathbf{p}$  be a limit point of  $X$ , let  $\mathbf{v}$  be a point of  $Y$ , let  $\varphi: X \rightarrow Y$  be a function mapping the set  $X$  into the set  $Y$ , and let  $\psi: Y \rightarrow \mathbb{R}^k$  be a function mapping the set  $Y$  into  $\mathbb{R}^k$ . Suppose that*

$$\lim_{\mathbf{x} \rightarrow \mathbf{p}} \varphi(\mathbf{x}) = \mathbf{v}$$

*and that the function  $\psi$  is continuous at  $\mathbf{v}$ . Then*

$$\lim_{\mathbf{x} \rightarrow \mathbf{p}} \psi(\varphi(\mathbf{x})) = \psi(\mathbf{v}).$$

**Proof**

Let some positive real number  $\varepsilon$  be given. Then there exists some positive real number  $\eta$  such that  $|\psi(\mathbf{y}) - \psi(\mathbf{v})| < \varepsilon$  for all  $\mathbf{y} \in Y$  satisfying  $|\mathbf{y} - \mathbf{v}| < \eta$ , because the function  $\psi$  is continuous at  $\mathbf{v}$ . But then there exists some positive real number  $\delta$  such that  $|\varphi(\mathbf{x}) - \mathbf{v}| < \eta$  for all  $\mathbf{x} \in X$  satisfying  $0 < |\mathbf{x} - \mathbf{p}| < \delta$ . It follows that  $|\psi(\varphi(\mathbf{x})) - \psi(\mathbf{v})| < \varepsilon$  for all  $\mathbf{x} \in X$  satisfying  $0 < |\mathbf{x} - \mathbf{p}| < \delta$ , and thus

$$\lim_{\mathbf{x} \rightarrow \mathbf{p}} \psi(\varphi(\mathbf{x})) = \psi(\mathbf{v}),$$

as required. ■

**Proposition 6.4**

Let  $X$  be a subset of  $\mathbb{R}^m$ , let  $f: X \rightarrow \mathbb{R}$  and  $g: X \rightarrow \mathbb{R}$  be real-valued functions on  $X$ , and let  $\mathbf{p}$  be a limit point of the set  $X$ . Suppose that  $\lim_{\mathbf{x} \rightarrow \mathbf{p}} f(\mathbf{x})$  and  $\lim_{\mathbf{x} \rightarrow \mathbf{p}} g(\mathbf{x})$  both exist. Then so do  $\lim_{\mathbf{x} \rightarrow \mathbf{p}} (f(\mathbf{x}) + g(\mathbf{x}))$ ,  $\lim_{\mathbf{x} \rightarrow \mathbf{p}} (f(\mathbf{x}) - g(\mathbf{x}))$  and  $\lim_{\mathbf{x} \rightarrow \mathbf{p}} (f(\mathbf{x})g(\mathbf{x}))$ , and moreover

$$\lim_{\mathbf{x} \rightarrow \mathbf{p}} (f(\mathbf{x}) + g(\mathbf{x})) = \lim_{\mathbf{x} \rightarrow \mathbf{p}} f(\mathbf{x}) + \lim_{\mathbf{x} \rightarrow \mathbf{p}} g(\mathbf{x}),$$

$$\lim_{\mathbf{x} \rightarrow \mathbf{p}} (f(\mathbf{x}) - g(\mathbf{x})) = \lim_{\mathbf{x} \rightarrow \mathbf{p}} f(\mathbf{x}) - \lim_{\mathbf{x} \rightarrow \mathbf{p}} g(\mathbf{x}),$$

$$\lim_{\mathbf{x} \rightarrow \mathbf{p}} (f(\mathbf{x})g(\mathbf{x})) = \lim_{\mathbf{x} \rightarrow \mathbf{p}} f(\mathbf{x}) \times \lim_{\mathbf{x} \rightarrow \mathbf{p}} g(\mathbf{x}),$$

If moreover  $g(\mathbf{x}) \neq 0$  for all  $\mathbf{x} \in X$  and  $\lim_{\mathbf{x} \rightarrow \mathbf{p}} g(\mathbf{x}) \neq 0$  then

$$\lim_{\mathbf{x} \rightarrow \mathbf{p}} \frac{f(\mathbf{x})}{g(\mathbf{x})} = \frac{\lim_{\mathbf{x} \rightarrow \mathbf{p}} f(\mathbf{x})}{\lim_{\mathbf{x} \rightarrow \mathbf{p}} g(\mathbf{x})}.$$

**Proof**

Let  $q = \lim_{\mathbf{x} \rightarrow \mathbf{p}} f(\mathbf{x})$  and  $r = \lim_{\mathbf{x} \rightarrow \mathbf{p}} g(\mathbf{x})$ , and let  $\psi: X \rightarrow \mathbb{R}^2$  be defined such that

$$\psi(\mathbf{x}) = (f(\mathbf{x}), g(\mathbf{x}))$$

for all  $\mathbf{x} \in X$ . Then

$$\lim_{\mathbf{x} \rightarrow \mathbf{p}} \psi(\mathbf{x}) = (q, r)$$

(see Proposition 6.1).

## 6. Limits of Functions of Several Real Variables (continued)

Let  $s: \mathbb{R}^2 \rightarrow \mathbb{R}$  and  $m: \mathbb{R}^2 \rightarrow \mathbb{R}$  be the functions from  $\mathbb{R}^2$  to  $\mathbb{R}$  defined such that  $s(u, v) = u + v$  and  $m(u, v) = uv$  for all  $u, v \in \mathbb{R}$ . Then the functions  $s$  and  $m$  are continuous (see Lemma 5.4). Also  $f + g = s \circ \psi$  and  $f \cdot g = m \circ \psi$ . It follows from this that

$$\begin{aligned}\lim_{\mathbf{x} \rightarrow \mathbf{p}} (f(\mathbf{x}) + g(\mathbf{x})) &= \lim_{\mathbf{x} \rightarrow \mathbf{p}} s(f(\mathbf{x}), g(\mathbf{x})) = \lim_{\mathbf{x} \rightarrow \mathbf{p}} s(\psi(\mathbf{x})) \\ &= s\left(\lim_{\mathbf{x} \rightarrow \mathbf{p}} \psi(\mathbf{x})\right) = s(q, r) = q + r,\end{aligned}$$

(see Lemma 6.3), and

$$\lim_{\mathbf{x} \rightarrow \mathbf{p}} (-g(\mathbf{x})) = -r.$$

It follows that

$$\lim_{\mathbf{x} \rightarrow \mathbf{p}} (f(\mathbf{x}) - g(\mathbf{x})) = q - r.$$



Similarly, when taking limits of products of functions,

$$\begin{aligned}\lim_{\mathbf{x} \rightarrow \mathbf{p}} (f(\mathbf{x})g(\mathbf{x})) &= \lim_{\mathbf{x} \rightarrow \mathbf{p}} m(f(\mathbf{x}), g(\mathbf{x})) = \lim_{\mathbf{x} \rightarrow \mathbf{p}} m(\psi(\mathbf{x})) \\ &= m\left(\lim_{\mathbf{x} \rightarrow \mathbf{p}} \psi(\mathbf{x})\right) = m(q, r) = qr\end{aligned}$$

## 6. Limits of Functions of Several Real Variables (continued)

Now suppose that  $g(\mathbf{x}) \neq 0$  for all  $\mathbf{x} \in X$  and that  $\lim_{\mathbf{x} \rightarrow \mathbf{p}} g(\mathbf{x}) \neq 0$ .

Representing the function sending  $\mathbf{x} \in X$  to  $1/g(\mathbf{x})$  as the composition of the function  $g$  and the reciprocal function  $e: \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}$ , where  $e(t) = 1/t$  for all non-zero real numbers  $t$ , we find, as in the first proof, that the function sending each point  $\mathbf{x}$  of  $X$  to

$$\lim_{\mathbf{x} \rightarrow \mathbf{p}} \left( \frac{1}{g(\mathbf{x})} \right) = \frac{1}{r}.$$

It then follows that

$$\lim_{\mathbf{x} \rightarrow \mathbf{p}} \frac{f(\mathbf{x})}{g(\mathbf{x})} = \frac{q}{r},$$

as required. ■

### 6.3. Relationships between Limits and Continuity

#### Proposition 6.5

*Let  $X$  be a subset of  $\mathbb{R}^m$ , let  $f: X \rightarrow \mathbb{R}^n$  be a function mapping the set  $X$  into  $\mathbb{R}^n$ , and let  $\mathbf{p}$  be a point of the set  $X$  that is also a limit point of  $X$ . Then the function  $f$  is continuous at the point  $\mathbf{p}$  if and only if  $\lim_{\mathbf{x} \rightarrow \mathbf{p}} f(\mathbf{x}) = f(\mathbf{p})$ .*

#### Proof

The result follows directly on comparing the relevant definitions. ■

Let  $X$  be a subset of  $m$ -dimensional Euclidean space  $\mathbb{R}^m$ , and let  $\mathbf{p}$  be a point of the set  $X$ . Suppose that the point  $\mathbf{p}$  is not a limit point of the set  $X$ . Then there exists some strictly positive real number  $\delta_0$  such that  $|\mathbf{x} - \mathbf{p}| \geq \delta_0$  for all  $\mathbf{x} \in X$  satisfying  $\mathbf{x} \neq \mathbf{p}$ . The point  $\mathbf{p}$  is then said to be an *isolated point* of  $X$ .

Let  $X$  be a subset of  $m$ -dimensional Euclidean space  $\mathbb{R}^m$ . The definition of continuity then ensures that any function  $\varphi: X \rightarrow \mathbb{R}^n$  mapping the set  $X$  into  $n$ -dimensional Euclidean space  $\mathbb{R}^n$  is continuous at any isolated point of its domain  $X$ .