

**MAU23203—Analysis in Several Variables**  
**School of Mathematics, Trinity College**  
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**Section 3: Convergence in Euclidean Spaces**

Trinity College Dublin

#### 3.1. Convergence of Infinite Sequences of Real Numbers

An *infinite sequence*  $x_1, x_2, x_3, \dots$  of real numbers associates to each positive integer  $j$  a corresponding real number  $x_j$ .

##### Definition

An infinite sequence  $x_1, x_2, x_3, \dots$  of real numbers is said to *converge* to some real number  $p$  if and only if the following criterion is satisfied:

*given any strictly positive real number  $\varepsilon$ , there exists some positive integer  $N$  such that  $|x_j - p| < \varepsilon$  for all positive integers  $j$  satisfying  $j \geq N$ .*

If an infinite sequence  $x_1, x_2, x_3, \dots$  of real numbers converges to some real number  $p$ , then  $p$  is said to be the *limit* of the sequence, and we can indicate the convergence of the infinite sequence to  $p$  by writing ' $x_j \rightarrow p$  as  $j \rightarrow +\infty$ ', or by writing ' $\lim_{j \rightarrow +\infty} x_j = p$ '.

### 3. Convergence in Euclidean Spaces (continued)

Let  $x$  and  $p$  be real numbers, and let  $\varepsilon$  be a strictly positive real number. Then  $|x - p| < \varepsilon$  if and only if both  $x - p < \varepsilon$  and  $p - x < \varepsilon$ . It follows that  $|x - p| < \varepsilon$  if and only if  $p - \varepsilon < x < p + \varepsilon$ . The condition  $|x - p| < \varepsilon$  essentially requires that the value of the real number  $x$  should agree with  $p$  to within an error of at most  $\varepsilon$ . An infinite sequence  $x_1, x_2, x_3, \dots$  of real numbers converges to some real number  $p$  if and only if, given any positive real number  $\varepsilon$ , there exists some positive integer  $N$  such that  $p - \varepsilon < x_j < p + \varepsilon$  for all positive integers  $j$  satisfying  $j \geq N$ .

### 3. Convergence in Euclidean Spaces (continued)

#### Definition

We say that an infinite sequence  $x_1, x_2, x_3, \dots$  of real numbers is *bounded above* if there exists some real number  $B$  such that  $x_j \leq B$  for all positive integers  $j$ . Similarly we say that this sequence is *bounded below* if there exists some real number  $A$  such that  $x_j \geq A$  for all positive integers  $j$ . A sequence is said to be *bounded* if it is bounded above and bounded below. Thus the sequence  $x_1, x_2, x_3, \dots$  is bounded if and only if there exist real numbers  $A$  and  $B$  such that  $A \leq x_j \leq B$  for all positive integers  $j$ .

#### Lemma 3.1

*Every convergent infinite sequence of real numbers is bounded.*

#### Proof

Let  $x_1, x_2, x_3, \dots$  be an infinite sequence of real numbers that converges to some real number  $p$ . On applying the formal definition of convergence (with  $\varepsilon = 1$ ), we deduce the existence of some positive integer  $N$  such that  $p - 1 < x_j < p + 1$  for all  $j \geq N$ . But then  $A \leq x_j \leq B$  for all positive integers  $j$ , where  $A$  is the minimum of  $x_1, x_2, \dots, x_{N-1}$  and  $p - 1$ , and  $B$  is the maximum of  $x_1, x_2, \dots, x_{N-1}$  and  $p + 1$ . ■

#### 3.2. Monotonic Sequences

An infinite sequence  $x_1, x_2, x_3, \dots$  of real numbers is said to be *strictly increasing* if  $x_{j+1} > x_j$  for all positive integers  $j$ , *strictly decreasing* if  $x_{j+1} < x_j$  for all positive integers  $j$ , *non-decreasing* if  $x_{j+1} \geq x_j$  for all positive integers  $j$ , *non-increasing* if  $x_{j+1} \leq x_j$  for all positive integers  $j$ . A sequence satisfying any one of these conditions is said to be *monotonic*; thus a monotonic sequence is either non-decreasing or non-increasing.

#### Theorem 3.2

*Any non-decreasing infinite sequence of real numbers that is bounded above is convergent. Similarly any non-increasing infinite sequence of real numbers that is bounded below is convergent.*

#### Proof

Let  $x_1, x_2, x_3, \dots$  be a non-decreasing sequence of real numbers that is bounded above. It follows from the Least Upper Bound Principle that there exists a least upper bound  $p$  for the set  $\{x_j : j \in \mathbb{N}\}$ . We claim that the sequence converges to  $p$ .

### 3. Convergence in Euclidean Spaces (continued)

Let some strictly positive real number  $\varepsilon$  be given. We must show that there exists some positive integer  $N$  such that  $|x_j - p| < \varepsilon$  whenever  $j \geq N$ . Now  $p - \varepsilon$  is not an upper bound for the set  $\{x_j : j \in \mathbb{N}\}$  (because  $p$  is the least upper bound), and therefore there must exist some positive integer  $N$  such that  $x_N > p - \varepsilon$ . But then  $p - \varepsilon < x_j \leq p$  whenever  $j \geq N$ , since the sequence is non-decreasing and bounded above by the real number  $p$ . Thus  $|x_j - p| < \varepsilon$  whenever  $j \geq N$ . Therefore  $x_j \rightarrow p$  as  $j \rightarrow +\infty$ , as required.

Next we note that if an infinite sequence  $x_1, x_2, x_3, \dots$  is non-increasing and bounded below then the sequence  $-x_1, -x_2, -x_3, \dots$  is non-decreasing and bounded above, and is therefore convergent. It follows that the sequence  $x_1, x_2, x_3, \dots$  is also convergent. ■



#### 3.3. Subsequences of Sequences of Real Numbers

##### Definition

Let  $x_1, x_2, x_3, \dots$  be an infinite sequence of real numbers. A *subsequence* of this infinite sequence is a sequence of the form  $x_{j_1}, x_{j_2}, x_{j_3}, \dots$  where  $j_1, j_2, j_3, \dots$  is an infinite sequence of positive integers with

$$j_1 < j_2 < j_3 < \dots$$

Let  $x_1, x_2, x_3, \dots$  be an infinite sequence of real numbers. The following sequences are examples of subsequences of this sequence:—

$$x_1, x_3, x_5, x_7, \dots$$

$$x_1, x_4, x_9, x_{16}, \dots$$

#### Theorem 3.3 (Bolzano-Weierstrass for the Real Line)

*Every bounded infinite sequence of real numbers has a convergent subsequence.*

#### Proof

Let some bounded infinite sequence  $x_1, x_2, x_3, \dots$  of real numbers be given. We define a *peak index* to be a positive integer  $j$  with the property that  $x_j \geq x_k$  for all positive integers  $k$  satisfying  $k \geq j$ . Thus a positive integer  $j$  is a peak index if and only if the  $j$ th member of the infinite sequence  $x_1, x_2, x_3, \dots$  is greater than or equal to all succeeding members of the sequence. Let  $S$  be the set consisting of all peak indices. Then

$$S = \{j \in \mathbb{N} : x_j \geq x_k \text{ for all } k \geq j\}.$$

### 3. Convergence in Euclidean Spaces (continued)

First let us suppose that the set of peak indices is infinite. Arrange the set  $S$  of peak indices in increasing order so that  $S = \{j_1, j_2, j_3, j_4, \dots\}$ , where  $j_1 < j_2 < j_3 < j_4 < \dots$ . It follows from the definition of peak indices that  $x_{j_1} \geq x_{j_2} \geq x_{j_3} \geq x_{j_4} \geq \dots$ . Thus  $x_{j_1}, x_{j_2}, x_{j_3}, \dots$  is a non-increasing subsequence of the given infinite sequence  $x_1, x_2, x_3, \dots$ . This subsequence is bounded below (since the given infinite sequence is bounded). It follows from Theorem 3.2 that  $x_{j_1}, x_{j_2}, x_{j_3}, \dots$  is a convergent subsequence of the given infinite sequence.

### 3. Convergence in Euclidean Spaces (continued)

Now suppose that the set  $S$  of peak indices is finite. Choose a positive integer  $j_1$  which is greater than every peak index. Then  $j_1$  is not a peak index. Therefore there must exist some positive integer  $j_2$  satisfying  $j_2 > j_1$  such that  $x_{j_2} > x_{j_1}$ . Moreover  $j_2$  is not a peak index (because  $j_2$  is greater than  $j_1$  and  $j_1$  in turn is greater than every peak index). Therefore there must exist some positive integer  $j_3$  satisfying  $j_3 > j_2$  such that  $x_{j_3} > x_{j_2}$ . We can continue in this fashion to construct (by induction on  $j$ ) a strictly increasing subsequence  $x_{j_1}, x_{j_2}, x_{j_3}, \dots$  of our original sequence. This increasing subsequence is bounded above (since the original sequence is bounded) and thus is convergent, by Theorem 3.2. This completes the proof of the one-dimensional case of the Bolzano-Weierstrass Theorem. ■

#### 3.5. Convergence of Sequences in Euclidean Spaces

##### Definition

An infinite sequence  $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \dots$  of points in  $\mathbb{R}^n$  is said to *converge* to a point  $\mathbf{p}$  if and only if, given strictly positive real number  $\varepsilon$ , there exists some positive integer  $N$  such that  $|\mathbf{x}_j - \mathbf{p}| < \varepsilon$  whenever  $j \geq N$ .

Given a convergent infinite sequence  $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \dots$  of points in  $\mathbb{R}^n$ , the point  $\mathbf{p}$  to which the sequence converges is referred to as the *limit* of the infinite sequence, and may be denoted by  $\lim_{j \rightarrow +\infty} \mathbf{x}_j$ .

### 3. Convergence in Euclidean Spaces (continued)

#### Lemma 3.4

*Let  $\mathbf{p}$  be a point of  $\mathbb{R}^n$ , where  $\mathbf{p} = (p_1, p_2, \dots, p_n)$ . Then an infinite sequence  $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \dots$  of points in  $\mathbb{R}^n$  converges to  $\mathbf{p}$  if and only if the  $i$ th components of the elements of this sequence converge to  $p_i$  for  $i = 1, 2, \dots, n$ .*

#### Proof

For each positive integer  $j$ , let  $(\mathbf{x}_j)_i$  denote the  $i$ th component of  $\mathbf{x}_j$ . Then  $|(\mathbf{x}_j)_i - p_i| \leq |\mathbf{x}_j - \mathbf{p}|$  for  $i = 1, 2, \dots, n$  and for all positive integers  $j$ . It follows directly from the definition of convergence that if  $\mathbf{x}_j \rightarrow \mathbf{p}$  as  $j \rightarrow +\infty$  then  $(\mathbf{x}_j)_i \rightarrow p_i$  as  $j \rightarrow +\infty$ .

### 3. Convergence in Euclidean Spaces (continued)

Conversely suppose that, for each integer  $i$  between 1 and  $n$ ,  $(\mathbf{x}_j)_i \rightarrow p_i$  as  $j \rightarrow +\infty$ . Let some positive real number  $\varepsilon$  be given. Then there exist positive integers  $N_1, N_2, \dots, N_n$  such that  $|(\mathbf{x}_j)_i - p_i| < \varepsilon/\sqrt{n}$  whenever  $j \geq N_i$ . Let  $N$  be the maximum of  $N_1, N_2, \dots, N_n$ . If  $j \geq N$  then  $j \geq N_i$  for  $i = 1, 2, \dots, n$ , and therefore

$$|\mathbf{x}_j - \mathbf{p}|^2 = \sum_{i=1}^n ((\mathbf{x}_j)_i - p_i)^2 < n \left( \frac{\varepsilon}{\sqrt{n}} \right)^2 = \varepsilon^2.$$

Thus  $\mathbf{x}_j \rightarrow \mathbf{p}$  as  $j \rightarrow +\infty$ , as required. ■

#### 3.6. The Multidimensional Bolzano-Weierstrass Theorem

##### **Theorem 3.5 (Multidimensional Bolzano-Weierstrass Theorem)**

*Every bounded sequence of points in a Euclidean space has a convergent subsequence.*



### 3. Convergence in Euclidean Spaces (continued)

#### **Proof**

The theorem is proved by induction on the dimension  $n$  of the space  $\mathbb{R}^n$  within which the points reside. When  $n = 1$ , the required result is the one-dimensional case of the Bolzano-Weierstrass Theorem, and the result has already been established in this case (see Theorem 3.3).

When  $n > 1$ , the result is proved in dimension  $n$  assuming the result in dimensions  $n - 1$  and 1. Consequently the result is established successively in dimensions 2, 3, 4,  $\dots$ , and therefore is valid for bounded sequences in  $\mathbb{R}^n$  for all positive integers  $n$ .

### 3. Convergence in Euclidean Spaces (continued)

It has been shown that every bounded infinite sequence of real numbers has a convergent subsequence (Theorem 3.3). Let  $n$  be an integer greater than one, and suppose, as an induction hypothesis, that, in cases where  $n > 2$ , all bounded sequences of points in  $\mathbb{R}^{n-1}$  have convergent subsequences. Let  $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \dots$  be a bounded infinite sequence in  $\mathbf{R}^n$  and, for each positive integer  $j$ , let  $\mathbf{s}_j$  denote the point of  $\mathbb{R}^{n-1}$  whose  $i$ th component is equal to the  $i$ th component  $x_{j,i}$  of  $\mathbf{x}_j$  for each integer  $i$  between 1 and  $n - 1$ .

### 3. Convergence in Euclidean Spaces (continued)

Let some strictly positive real number  $\varepsilon$  be given. Now the infinite sequence

$$\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3, \dots$$

of points of  $\mathbb{R}^{n-1}$  is a bounded infinite sequence. In the case when  $n = 2$  we can apply the one-dimensional Bolzano-Weierstrass Theorem (Theorem 3.3) to conclude that this sequence of real numbers has a convergent subsequence. In cases where  $n > 2$ , we are supposing as our induction hypothesis that any bounded sequence in  $\mathbb{R}^{n-1}$  has a convergent subsequence. Thus, assuming this induction hypothesis in cases where  $n > 2$ , we can conclude, in all cases with  $n > 1$ , that the bounded infinite sequence  $\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3, \dots$  of points in  $\mathbb{R}^{n-1}$  has a convergent subsequence.

### 3. Convergence in Euclidean Spaces (continued)

Let that convergent subsequence be

$$\mathbf{s}_{m_1}, \mathbf{s}_{m_2}, \mathbf{s}_{m_3}, \dots,$$

where  $m_1, m_2, m_3, \dots$  is a strictly increasing infinite sequence of positive integers, and let  $\mathbf{q} = \lim_{j \rightarrow +\infty} \mathbf{s}_{m_j}$ . There then exists some positive integer  $L$  such that

$$|\mathbf{s}_{m_j} - \mathbf{q}| < \frac{1}{2}\varepsilon$$

for all positive integers  $j$  for which  $m_j \geq L$ . (Indeed the definition of convergence ensures the existence of a positive integer  $N$  that is large enough to ensure that  $|\mathbf{s}_{m_j} - \mathbf{q}| < \frac{1}{2}\varepsilon$  whenever  $j \geq N$ . Taking  $L = m_N$  then ensures that  $j \geq N$  whenever  $m_j \geq L$ .)

### 3. Convergence in Euclidean Spaces (continued)

Let  $t_j$  denote the  $n$ th component of the point  $\mathbf{x}_j$  of  $\mathbb{R}^n$  for each positive integer  $j$ . The one-dimensional Bolzano-Weierstrass Theorem ensures that the bounded infinite sequence

$$t_{m_1}, t_{m_2}, t_{m_3}, \dots$$

of real numbers has a convergent subsequence. It follows that there is a strictly increasing infinite sequence  $k_1, k_2, k_3, \dots$  of positive integers, where each  $k_j$  is equal to one of the positive integers  $m_1, m_2, m_3, \dots$ , such that the infinite sequence

$$t_{k_1}, t_{k_2}, t_{k_3}, \dots$$

is convergent.

### 3. Convergence in Euclidean Spaces (continued)

Let  $r = \lim_{j \rightarrow +\infty} t_{k_j}$ . There then exists some positive integer  $M$  such that  $M \geq L$  and

$$|t_{k_j} - r| < \frac{1}{2}\varepsilon$$

for all positive integers  $j$  for which  $k_j \geq M$ . It follows that if  $k_j \geq M$  then

$$|\mathbf{s}_{k_j} - \mathbf{q}| < \frac{1}{2}\varepsilon \quad \text{and} \quad |t_{k_j} - r| < \frac{1}{2}\varepsilon.$$

Now there is a point  $\mathbf{p}$  of  $\mathbb{R}^n$ , where  $\mathbf{p} = (p_1, p_2, \dots, p_n)$ , determined so that the  $i$ th components of the point  $\mathbf{p}$  of  $\mathbb{R}^n$  is equal to the  $i$ th component of the point  $\mathbf{q}$  of  $\mathbb{R}^{n-1}$  for each integer  $i$  between 1 and  $n-1$  and also the  $n$ th component of the point  $\mathbf{p}$  is equal to the real number  $t$ .

### 3. Convergence in Euclidean Spaces (continued)

Also it follows from the definition of the Euclidean norm that

$$|\mathbf{x}_{k_j} - \mathbf{p}|^2 = |\mathbf{s}_{k_j} - \mathbf{q}|^2 + |t_{k_j} - r|^2 < \frac{1}{2}\varepsilon^2$$

whenever  $k_j \geq M$ . But then  $|\mathbf{x}_{k_j} - \mathbf{p}| < \varepsilon$  for all positive integers  $j$  for which  $k_j \geq M$ . It follows that  $\lim_{j \rightarrow +\infty} \mathbf{x}_{k_j} = \mathbf{p}$ . We conclude therefore that the bounded infinite sequence  $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \dots$  does indeed have a convergent subsequence. This completes the proof of the Bolzano-Weierstrass Theorem in dimension  $n$  for all positive integers  $n$ . ■

#### 3.7. Cauchy Sequences in Euclidean Spaces

##### Definition

An infinite sequence  $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \dots$  of points of  $n$ -dimensional Euclidean space  $\mathbb{R}^n$  is said to be a *Cauchy sequence* if, given any strictly positive real number  $\varepsilon$ , there exists some positive integer  $N$  such that  $|\mathbf{x}_j - \mathbf{x}_k| < \varepsilon$  for all positive integers  $j$  and  $k$  satisfying  $j \geq N$  and  $k \geq N$ .



#### Lemma 3.6

*Every Cauchy sequence of points of  $n$ -dimensional Euclidean space  $\mathbb{R}^n$  is bounded.*

#### Proof

Let  $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \dots$  be a Cauchy sequence of points in  $\mathbb{R}^n$ . Then there exists some positive integer  $N$  such that  $|\mathbf{x}_j - \mathbf{x}_k| < 1$  whenever  $j \geq N$  and  $k \geq N$ . In particular,  $|\mathbf{x}_j| \leq |\mathbf{x}_N| + 1$  whenever  $j \geq N$ . Therefore  $|\mathbf{x}_j| \leq R$  for all positive integers  $j$ , where  $R$  is the maximum of the real numbers  $|\mathbf{x}_1|, |\mathbf{x}_2|, \dots, |\mathbf{x}_{N-1}|$  and  $|\mathbf{x}_N| + 1$ . Thus the sequence is bounded, as required. ■

### 3. Convergence in Euclidean Spaces (continued)

#### Theorem 3.7 (Cauchy's Criterion for Convergence)

*An infinite sequence of points of  $n$ -dimensional Euclidean space  $\mathbb{R}^n$  is convergent if and only if it is a Cauchy sequence.*

#### Proof

First we show that convergent sequences in  $\mathbb{R}^n$  are Cauchy sequences. Let  $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \dots$  be a convergent sequence of points in  $\mathbb{R}^n$ , and let  $\mathbf{p} = \lim_{j \rightarrow +\infty} \mathbf{x}_j$ . Let some strictly positive real number  $\varepsilon$  be given. Then there exists some positive integer  $N$  such that  $|\mathbf{x}_j - \mathbf{p}| < \frac{1}{2}\varepsilon$  for all  $j \geq N$ . Thus if  $j \geq N$  and  $k \geq N$  then  $|\mathbf{x}_j - \mathbf{p}| < \frac{1}{2}\varepsilon$  and  $|\mathbf{x}_k - \mathbf{p}| < \frac{1}{2}\varepsilon$ , and hence

$$|\mathbf{x}_j - \mathbf{x}_k| = |(\mathbf{x}_j - \mathbf{p}) - (\mathbf{x}_k - \mathbf{p})| \leq |\mathbf{x}_j - \mathbf{p}| + |\mathbf{x}_k - \mathbf{p}| < \varepsilon.$$

Thus the sequence  $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \dots$  is a Cauchy sequence.

### 3. Convergence in Euclidean Spaces (continued)

Conversely we must show that any Cauchy sequence  $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \dots$  in  $\mathbb{R}^n$  is convergent. Now Cauchy sequences are bounded, by Lemma 3.6. The sequence  $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \dots$  therefore has a convergent subsequence  $\mathbf{x}_{k_1}, \mathbf{x}_{k_2}, \mathbf{x}_{k_3}, \dots$ , by the multidimensional Bolzano-Weierstrass Theorem (Theorem 3.5). Let  $\mathbf{p} = \lim_{j \rightarrow +\infty} \mathbf{x}_{k_j}$ . We claim that the sequence  $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \dots$  itself converges to  $\mathbf{p}$ .

### 3. Convergence in Euclidean Spaces (continued)

Let some strictly positive real number  $\varepsilon$  be given. Then there exists some positive integer  $N$  such that  $|\mathbf{x}_j - \mathbf{x}_k| < \frac{1}{2}\varepsilon$  whenever  $j \geq N$  and  $k \geq N$  (since the sequence is a Cauchy sequence). Let  $m$  be chosen large enough to ensure that  $k_m \geq N$  and  $|\mathbf{x}_{k_m} - \mathbf{p}| < \frac{1}{2}\varepsilon$ . Then

$$|\mathbf{x}_j - \mathbf{p}| \leq |\mathbf{x}_j - \mathbf{x}_{k_m}| + |\mathbf{x}_{k_m} - \mathbf{p}| < \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon$$

whenever  $j \geq N$ . It follows that  $\mathbf{x}_j \rightarrow \mathbf{p}$  as  $j \rightarrow +\infty$ , as required. ■