

MAU23203: Analysis in Several Real Variables
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Disquisition IV: Further Examples concerning
Open And Closed Sets

David R. Wilkins

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Example Consider the set

$$\{(x, y, z) \in \mathbb{R}^3 : \sin(x^2 + y^2) < \cos(y^2 + z^2)\}.$$

This set is open. Indeed function $f: \mathbb{R}^3 \rightarrow \mathbb{R}$ defined such that

$$f(x, y, z) = \cos(y^2 + z^2) - \sin(x^2 + y^2)$$

for all $(x, y, z) \in \mathbb{R}^3$ is continuous, and the given set is the preimage of the set of positive real numbers under this function, and thus is the preimage of an open set under a continuous function. The set is not closed. The point $(0, \frac{1}{2}\sqrt{\pi}, 0)$ does not belong to the set, but every open ball of positive radius about this point contains points $(0, y, 0)$ where $y < \frac{1}{2}\sqrt{\pi}$, and such points do not belong to the set.

We conclude therefore that the set

$$\{(x, y, z) \in \mathbb{R}^3 : \sin(x^2 + y^2) < \cos(y^2 + z^2)\}.$$

is open, and not closed.

Example Consider the set

$$\{(x, y, z) \in \mathbb{R}^3 : z > 0 \text{ and } z(x^2 + y^2) = 1\}.$$

This set is closed. Indeed can be written in the form

$$\{(x, y, z) \in \mathbb{R}^3 : z \geq 0 \text{ and } z(x^2 + y^2) = 1\}.$$

The set $\{(x, y, z) \in \mathbb{R}^3 : z \geq 0\}$ is

$$\{(x, y, z) \in \mathbb{R}^3 : z(x^2 + y^2) = 1\}$$

is also a closed set as it is the preimage of the closed set $\{1\}$ under a continuous function of x , y and z . The intersection of these two closed sets is the given set. The set is not open: the point $(1, 0, 1)$ belongs to the set, but no open ball of positive radius about this point is contained in the set.

We conclude therefore that the set

$$\{(x, y, z) \in \mathbb{R}^3 : z > 0 \text{ and } z(x^2 + y^2) = 1\}.$$

is closed, and not open.

Example Let $g: \mathbb{R} \rightarrow \mathbb{R}$ and $h: \mathbb{R} \rightarrow \mathbb{R}$ be continuous real-valued functions on \mathbb{R} . Consider the set

$$\{(x, y) \in \mathbb{R}^2 : g(x) < y < h(x)\}$$

We show that this set is open in \mathbb{R}^2 . To this end, let $\varphi: \mathbb{R}^2 \rightarrow \mathbb{R}$ and $\psi: \mathbb{R}^2 \rightarrow \mathbb{R}$ be defined so that $\varphi(x, y) = y - g(x)$ and $\psi(x, y) = h(x) - y$ for all $(x, y) \in \mathbb{R}^2$. Also let $P = \{t \in \mathbb{R} : t > 0\}$. Then P is open in \mathbb{R} . Now the set X in question satisfies $X = X_1 \cap X_2$, where

$$\begin{aligned} X_1 &= \{(x, y) \in \mathbb{R}^2 : g(x) < y\} = \{(x, y) \in \mathbb{R}^2 : \varphi(x, y) > 0\} = \varphi^{-1}(P), \\ X_2 &= \{(x, y) \in \mathbb{R}^2 : y < h(x)\} = \{(x, y) \in \mathbb{R}^2 : \psi(x, y) > 0\} = \psi^{-1}(P), \end{aligned}$$

It follows from Proposition 5.7 that the sets X_1 and X_2 are open in \mathbb{R}^2 . Thus the given set X is the intersection of two open sets, and is thus itself open in \mathbb{R}^2 .

Example Consider the subset Y of \mathbb{R}^2 defined such that

$$Y = \{(x, y) \in \mathbb{R}^2 : \text{there exists } n \in \mathbb{Z} \text{ such that } (x - n)^2 + y^2 < 1\}.$$

(In other words, a point (x, y) of \mathbb{R}^2 belongs to Y if and only if some integer n , depending on the values of x and y , can be found for which $(x - n)^2 + y^2 < 1$.)

The set Y is open in \mathbb{R}^2 . It is the union of open balls centred on the points $(n, 0)$ as n ranges over the set \mathbb{Z} of integers. Any open ball is an open set, and any union of open sets is an open set. Therefore the set Y is an open set in \mathbb{R}^2 .

The set Y is not closed in \mathbb{R}^2 . The point $(0, 1)$ belongs to the complement of the set Y , but any open ball of radius δ about this point contains points $(0, u)$ of \mathbb{R}^2 with $0 \leq u < 1$ and $u > 1 - \delta$, and such points do belong to the set Y . Thus no open ball of positive radius centred on the point $(0, 1)$ is contained within the complement of the set Y . Thus the complement $\mathbb{R}^2 \setminus Y$ of Y is not open, and therefore the set Y itself is not closed.

Example Consider the subset Z of \mathbb{R}^2 defined such that

$$Z = \{(x, y) \in \mathbb{R}^2 : \text{there exists } n \in \mathbb{Z} \text{ such that } (x - n)^2 + y^2 \leq 1\}.$$

The set Z is not open in \mathbb{R}^2 . Indeed point $(0, 1)$ belongs to the set Z , but any open ball of radius δ about this point contains points $(0, u)$ of \mathbb{R}^2 with $0 \leq u < 1$ and $u > 1 - \delta$, and such points do not belong to the set Z . Thus no open ball of positive radius centred on the point $(0, 1)$ is contained within the set Z .

The set Z is closed in \mathbb{R}^2 . Let (u, v) be a point of the complement of Z . An integer M can be found large enough to ensure that $-M + 2 < u < M - 2$. If n is an integer satisfying $|n| \geq M$ then the closed ball of radius 1 about the point $(n, 0)$ does not intersect the open ball of radius 1 about the point (u, v) . Let F be the union of the closed balls of radius 1 about the points $(n, 0)$ for which n is an integer satisfying $|n| < M$. The set F is then a finite union of closed balls. Moreover closed balls are closed sets, and any finite union of closed sets is closed. It follows that F is a closed set. Therefore some real number δ satisfying $0 < \delta < 1$ can be found so that the open ball of radius δ about (u, v) does not intersect the set F . Then the open ball of radius δ about (u, v) does not intersect the set Z . This shows that the complement of the set Z is open, and therefore the set Z itself is closed in \mathbb{R}^2 .

Proposition A *A subset X of \mathbb{R}^n is closed in \mathbb{R}^n if and only if the limit of every convergent sequence of points in X belongs to X .*

Proof First let X be a closed set in \mathbb{R}^n . Then the limit of every convergent sequence of points of X belongs to the set X (see Lemma 4.7).

Now let X be a subset of \mathbb{R}^n with the property that, for every convergent sequence of points of X , the limit of that sequence belongs to the set X . Let \mathbf{p} be a point of \mathbb{R}^n that does not belong to the set X . Suppose that the complement of the set \mathbb{R}^n were not an open set. Suppose that there did not exist any positive number δ with the property that the open ball of radius δ centred on the point \mathbf{p} was wholly contained within the complement of the set X . Then every open ball of positive radius centered on the point \mathbf{p} would have non-empty intersection with the set X , and therefore

there would exist an infinite sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \dots$ of points of the set X with the property that $|\mathbf{x}_j - \mathbf{p}| < 1/j$ for all positive integers j . This infinite sequence would converge to the point \mathbf{p} . But the point \mathbf{p} has been chosen so that it lies outside the set X , and the set X in question has the property that every convergent infinite sequence within it converges to a point of the set X . Thus a contradiction would arise unless there exists some strictly positive real number δ with the property that the open ball of radius δ centred on the point \mathbf{p} is wholly contained within the complement of the set X . This argument establishes that the complement of the set X is open in \mathbb{R}^n , because, given any point of the complement of X , there exists some open ball of positive radius centred on that point that is wholly contained within the complement of the set X . Consequently the set X is closed in \mathbb{R}^n , as required. ■

Example We resume discussion of the subset Z of \mathbb{R}^2 defined so that

$$Z = \{(x, y) \in \mathbb{R}^2 : \text{there exists } n \in \mathbb{Z} \text{ such that } (x - n)^2 + y^2 \leq 1\}.$$

Let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \dots$ be a convergent sequence of points in Z , and let \mathbf{p} be the limit of this sequence. We show that $\mathbf{p} \in Z$. Now the infinite sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \dots$ is bounded, because all convergent sequences are bounded.

Therefore there exists a positive integer M such that $\mathbf{x}_j \in \bigcup_{n=-M}^M \overline{B}((n, 0), 1)$, where $\overline{B}((n, 0), 1)$ denotes the closed ball of radius 1 about the point $(n, 0)$.

Now closed balls are closed sets, and therefore $\bigcup_{n=-M}^M \overline{B}((n, 0), 1)$, being a finite union of closed sets must itself be a closed set. It follows from this that $\mathbf{p} \in \bigcup_{n=-M}^M \overline{B}((n, 0), 1)$, and therefore $\mathbf{p} \in Z$. We conclude therefore that the limit of every convergent sequence of points belonging to Z must itself belong to the set Z . Applying Proposition A above, we conclude that (as was shown above by a different method) the set Z is closed in \mathbb{R}^2 .