MAU23203: Analysis in Several Real Variables
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Disquisition VIII: Examples of Differentiability and Non-Differentiability

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Example Consider the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ that is defined such that $f(x,y) = \min(|x|,|y|)$ for all $(x,y) \in \mathbb{R}^2$.

The function f is continuous at $(0,0)$. Inded $|f(x,y)| \leq \sqrt{x^2+y^2}$ for all $(x,y) \in \mathbb{R}^2$. Let some positive real number ε be given. If $|(x,y)| < \varepsilon$ then $|f(x,y)| < \varepsilon$. Thus the definition of continuity is satisfied at $(x,y) = 0$.

The function f is not differentiable at $(0,0)$. Note that

$$\left.\frac{\partial f}{\partial x}\right|_{(0,0)} = 0 \quad \text{and} \quad \left.\frac{\partial f}{\partial y}\right|_{(0,0)} = 0.$$

If it were the case that the function were differentiable at zero, then the derivative of the function at $(0,0)$ would be determined by the above partial derivatives, and would therefore be zero. It would then follow that

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y)}{\sqrt{x^2+y^2}} = 0.$$

Suppose that $x = y = t$. Then $f(x,y) = |t|$ and $\sqrt{x^2+y^2} = \sqrt{2}\,t$. It follows that

$$\lim_{t\to 0+} \frac{f(t,t)}{\sqrt{t^2+t^2}} = \frac{1}{\sqrt{2}}.$$

Thus it cannot be the case that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{f(x,y)}{\sqrt{x^2+y^2}} = 0$. Therefore the function f is not differentiable at $(0,0)$.

Example Consider the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ that is defined such that $f(x,y) = \min(x^2, y^2)$ for all $(x,y) \in \mathbb{R}^2$.

This function is continuous and differentiable at $(0,0)$. Note that $f(x,y) \leq x^2 + y^2$ for all $(x,y) \in \mathbb{R}^2$, and therefore

$$\frac{|f(x,y)|}{\sqrt{x^2+y^2}} \leq \sqrt{x^2+y^2}$$

for all $(x,y) \in \mathbb{R}^2$. It follows that

$$\lim_{(x,y)\to(0,0)} \frac{|f(x,y)|}{\sqrt{x^2+y^2}} = 0.$$

It then follows from the definition of differentiability that that function f is differentiable at $(0,0)$, and its derivative at $(0,0)$ is zero. Differentiability implies continuity. The function f is thus continuous at $(0,0)$.

Example Consider the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ defined so that

$$f(x,y) = \begin{cases} \dfrac{x^3+y^3}{x^2+y^2} & \text{if } (x,y) \neq (0,0); \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

It follows from straightforward applications of the Product and Chain Rules for functions of several real variables that the function f is differentiable at each point of $\mathbb{R}^2 \setminus \{(0,0)\}$. This result also follows from the fact that the first order partial derivatives of the function f are defined and continuous throughout the set $\mathbb{R}^2 \setminus \{(0,0)\}$. Indeed calculating the first order partial derivatives of the function f away from the origin, we find that

$$\frac{\partial f}{\partial x} = \frac{x^4 + 3x^2y^2 - 2xy^3}{(x^2+y^2)^2} \quad \text{and} \quad \frac{\partial f}{\partial y} = \frac{y^4 + 3x^2y^2 - 2x^3y}{(x^2+y^2)^2}$$

when $(x,y) \neq (0,0)$. Thus, away from the origin $(0,0)$, the first order partial derivatives of f are quotients of continuous functions, and must therefore themselves be continuous functions.

The function f itself is continuous at $(0,0)$. Indeed $|x^3| \leq (\sqrt{x^2+y^2})^3$ and $|y^3| \leq (\sqrt{x^2+y^2})^3$ for all $(x,y) \in \mathbb{R}^2$, and therefore $|f(x,y)| \leq 2\sqrt{x^2+y^2}$ for all $(x,y) \in \mathbb{R}^2$. Thus, given any positive real number ε, the inequality $|f(x,y)| < \varepsilon$ is satisfied whenever the point (x,y) lies within a distance $\frac{1}{2}\varepsilon$ of the origin $(0,0)$.

Also

$$\left.\frac{\partial f}{\partial x}\right|_{(x,y)=(0,0)} = 1 \quad \text{and} \quad \left.\frac{\partial f}{\partial y}\right|_{(x,y)=(0,0)} = 1.$$

Now let b and c be real numbers, not both zero, and let $u_{b,c}(t) = f(bt, ct)$ for all real numbers t. Then

$$u_{b,c}(t) = \frac{b^3 + c^3}{b^2 + c^2}\, t$$

for all real numbers t, and therefore

$$\frac{d}{dt}\left(u_{b,c}(t)\right) = \frac{b^3 + c^3}{b^2 + c^2}$$

for all real numbers t. Now if it were the case that the function f was differentiable at $(0, 0)$, it would follow on applying the Chain Rule for differentiable functions of several real variables that

$$\begin{aligned} \left.\frac{d}{dt}\left(u_{b,c}(t)\right)\right|_{t=0} &= b\left.\frac{\partial f}{\partial x}\right|_{(x,y)=(0,0)} + c\left.\frac{\partial f}{\partial y}\right|_{(x,y)=(0,0)} \\ &= b + c \end{aligned}$$

for all real numbers b and c that were not both zero. However the equation

$$\frac{b^3 + c^3}{b^2 + c^2} = b + c$$

is satisfied if and only if $bc(b + c) = 0$. It follows that the function f is *not* differentiable at $(0, 0)$.

Note also that

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = \tfrac{1}{2} \quad \text{whenever} \quad x = y \text{ and } (x, y) \neq (0, 0).$$

But thes partial derivatives have the value 1 when $(x, y) = (0, 0)$. Thus the first order partial derivatives of the function f are not continuous at the origin $(0, 0)$.

Example Consider the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ defined so that

$$f(x, y) = \begin{cases} \dfrac{xy}{(x^2 + y^2)^2} & \text{if } (x, y) \neq (0, 0); \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

Note that this function is not continuous at $(0, 0)$. Indeed $f(t, t) = 1/(4t^2)$ if $t \neq 0$ so that $f(t, t) \to +\infty$ as $t \to 0$, yet $f(x, 0) = f(0, y) = 0$ for all $x, y \in \mathbb{R}$, thus showing that

$$\lim_{(x,y)\to(0,0)} f(x, y)$$

cannot possibly exist. Because f is not continuous at $(0,0)$ we conclude from Lemma 8.11 that f cannot be differentiable at $(0,0)$. However it is easy to show that the partial derivatives

$$\frac{\partial f(x,y)}{\partial x} \text{ and } \frac{\partial f(x,y)}{\partial y}$$

exist everywhere on $\mathbb{R}^2$, even at $(0,0)$. Indeed

$$\left.\frac{\partial f(x,y)}{\partial x}\right|_{(x,y)=(0,0)} = 0, \qquad \left.\frac{\partial f(x,y)}{\partial y}\right|_{(x,y)=(0,0)} = 0$$

on account of the fact that $f(x,0) = f(0,y) = 0$ for all $x, y \in \mathbb{R}$.

Example Consider the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ defined so that

$$f(x,y) = \begin{cases} \dfrac{xy^2}{x^2+y^4} & \text{if } (x,y) \neq (0,0); \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Given real numbers b and c, let $u_{b,c}\colon \mathbb{R} \to \mathbb{R}$ be defined so that $u_{b,c}(t) = f(bt,ct)$ for all $t \in \mathbb{R}$. If $b = 0$ or $c = 0$ then $u_{b,c}(t) = 0$ for all $t \in \mathbb{R}$. If $b \neq 0$ and $c \neq 0$ then

$$u_{b,c}(t) = \frac{bc^2t^3}{b^2t^2 + c^4t^4} = \frac{bc^2t}{b^2 + c^2t^2}.$$

We now show that the function $u_{b,c}\colon \mathbb{R} \to \mathbb{R}$ has derivatives of all orders. This is obvious when $b = 0$, and when $c = 0$. If b and c are both non-zero, and if the function $u_{b,c}$ has a derivative $u_{b,c}^{(k)}(t)$ of order k that can be represented in the form

$$u_{b,c}^{(k)}(t) = p_k(t)(b^2 + c^2t^2)^{-k-1},$$

where $p_k(t)$ is a polynomial of degree at most $k+1$, then it follows from standard single-variable calculus that the function $u_{b,c}$ has a derivative $u_{b,c}^{(k+1)}(t)$ of order $k+1$ that can be represented in the form

$$u_{b,c}^{(k+1)}(t) = p_{k+1}(t)(b^2 + c^2t^2)^{-k-2},$$

where $p_{k+1}(t)$ is the polynomial of degree at most $k+2$ determined by the formula

$$p_{k+1}(t) = p_k'(t)(b^2 + c^2t^2) - 2(k+1)c^2tp_k(t).$$

Thus the function $u_{b,c}\colon \mathbb{R} \to \mathbb{R}$ has derivatives of all orders.

Moreover the first derivative $u'_{b,c}(0)$ of $u_{b,c}(t)$ at $t = 0$ is given by the formula

$$u'_{b,c}(0) = \begin{cases} \dfrac{c^2}{b} & \text{if } b \neq 0; \\ 0 & \text{if } b = 0. \end{cases}$$

We have shown that the restriction of the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ to any line passing through the origin determines a function that may be differentiated any number of times with respect to distance along the line. Analogous arguments show that the restriction of the function g to any other line in the plane also determines a function that may be differentiated any number of times with respect to distance along the line.

Now $f(x, y) = \frac{1}{2}$ for all $(x, y) \in \mathbb{R}^2$ satisfying $x > 0$ and $y = \pm\sqrt{x}$, and similarly $f(x, y) = -\frac{1}{2}$ for all $(x, y) \in \mathbb{R}^2$ satisfying $x < 0$ and $y = \pm\sqrt{-x}$. It follows that every open disk about the origin $(0, 0)$ contains some points at which the function f takes the value $\frac{1}{2}$, and other points at which the function takes the value $-\frac{1}{2}$, and indeed the function f will take on all real values between $-\frac{1}{2}$ and $\frac{1}{2}$ on any open disk about the origin, no matter how small the disk. Therefore the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ is not continuous at zero, even though the partial derivatives of the function f with respect to x and y exist at each point of $\mathbb{R}^2$.

Remark Examination of some of the examples discussed above establishes that *even if all the partial derivatives of a function exist at some point, this does not necessarily imply that the function is differentiable at that point.* However it is a standard result in the theory of differentiability for functions of several real variables that if the first order partial derivatives of the components of a function exist *and are continuous* throughout some neighbourhood of a given point then the function is differentiable at that point (see Proposition 8.12).