MAU23203—Analysis in Several Variables
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Section 4: Open and Closed Sets in Euclidean Spaces

Trinity College Dublin

4. Open and Closed Sets in Euclidean Spaces

4.1. Open Sets in Euclidean Spaces

Definition

Given a point $\mathbf{p}$ of $\mathbb{R}^n$ and a positive real number η, the *open ball* $B(\mathbf{p}, \eta)$ in $\mathbb{R}^n$ of *radius* η centred on the point $\mathbf{p}$ consists of all points of $\mathbb{R}^n$ whose Euclidean distance from the point $\mathbf{p}$ is less than η.

We see therefore that

$$B(\mathbf{p}, \eta) = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x} - \mathbf{p}| < \eta\}$$

for all points $\mathbf{p}$ of $\mathbb{R}^n$ and positive real numbers η.

The *open ball* $B(\mathbf{p}, \eta)$ of radius η centred on a point $\mathbf{p}$ of $\mathbb{R}^n$ is bounded by the *sphere* of radius η centred on $\mathbf{p}$. This sphere is the set

$$\{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x} - \mathbf{p}| = \eta\}.$$

Definition

A subset V of $\mathbb{R}^n$ is said to be an *open set* (in $\mathbb{R}^n$) if, given any point of V, there exists an open ball of positive radius, centred on that point, which is wholly contained within the set V.

By convention the empty set $\emptyset$ is also considered to be an open set (on the grounds that there does not exist any point of the empty set that is not the centre of some open ball contained in the empty set).

Thus a subset V of $\mathbb{R}^n$ is an open set in $\mathbb{R}^n$ if and only if, given any point $\mathbf{p}$ of V, there exists some strictly positive real number δ such that $B(\mathbf{p}, \delta) \subset V$, where

$$B(\mathbf{p}, \delta) = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x} - \mathbf{p}| < \delta\}.$$

Example

Let $H = \{(x, y, z) \in \mathbb{R}^3 : z > c\}$, where c is some real number. Then H is an open set in $\mathbb{R}^3$. Indeed let $\mathbf{p}$ be a point of H. Then $\mathbf{p} = (u, v, w)$, where $w > c$. Let $\delta = w - c$. If the distance from a point (x, y, z) to the point (u, v, w) is less than δ then $|z - w| < \delta$, and hence $z > c$, so that $(x, y, z) \in H$. Thus $B(\mathbf{p}, \delta) \subset H$, and therefore H is an open set.

The previous example can be generalized. Given any integer i between 1 and n, and given any real number c_i, the sets

$$\{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_i > c_i\}$$

and

$$\{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_i < c_i\}$$

are open sets in $\mathbb{R}^n$.

Example

Let

$$V = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 < 9\}.$$

Then the subset V of $\mathbb{R}^3$ is the open ball of radius 3 in $\mathbb{R}^3$ centred on the origin. This open ball is an open set. Indeed let $\mathbf{q}$ be a point of V. Then $|\mathbf{q}| < 3$. Let $\delta = 3 - |\mathbf{q}|$. Then $\delta > 0$. Moreover if $\mathbf{x}$ is a point of $\mathbb{R}^3$ that satisfies $|\mathbf{x} - \mathbf{q}| < \delta$ then

$$|\mathbf{x}| = |\mathbf{q} + (\mathbf{x} - \mathbf{q})| \leq |\mathbf{q}| + |\mathbf{x} - \mathbf{q}| < |\mathbf{q}| + \delta = 3,$$

and therefore $\mathbf{x} \in V$. This proves that V is an open set.

More generally, an open ball of any positive radius centred on any point of a Euclidean space $\mathbb{R}^n$ of any dimension n is an open set in that Euclidean space. A more general result is proved below (see Lemma 4.1).

4.2. Open Sets in Subsets of Euclidean Spaces

Definition

Let X be a subset of n-dimensional Euclidean space $\mathbb{R}^n$. Given a point $\mathbf{p}$ of X and a positive real number η, the *open ball* $B_X(\mathbf{p}, \eta)$ in X of *radius* η centred on the point $\mathbf{p}$ consists of all points of the set X whose Euclidean distance from the point $\mathbf{p}$ is less than η.

We see therefore that

$$B_X(\mathbf{p}, \eta) = \{\mathbf{x} \in X : |\mathbf{x} - \mathbf{p}| < \eta\}$$

for all points $\mathbf{p}$ of X and positive real numbers η.

Definition

Let X be a subset of n-dimensional Euclidean space $\mathbb{R}^n$. A subset V of X is said to be *open in* X if, given any point of V, there exists an open ball in X of positive radius, centred on that point, which is wholly contained within the set V.

By convention the empty set $\emptyset$ is also considered to be open in the given set X (on the grounds that there does not exist any point of the empty set that is not the centre of some open ball contained in the empty set).

Thus given any subset X of $\mathbb{R}^n$, and given any subset V of X, the set V is said to be open in X if and only if, given any point $\mathbf{p}$ of V, there exists some strictly positive real number δ such that $B_X(\mathbf{p}, \delta) \subset V$, where

$$B_X(\mathbf{p}, \delta) = \{\mathbf{x} \in X : |\mathbf{x} - \mathbf{p}| < \delta\}.$$

Example

Let V be an open set in $\mathbb{R}^n$. Then for any subset X of $\mathbb{R}^n$, the intersection $V \cap X$ is open in X. (This follows directly from the definitions.) Thus for example, let S^2 be the unit sphere in $\mathbb{R}^3$, given by

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$$

and let N be the subset of S^2 given by

$$N = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1 \text{ and } z > 0\}.$$

Then N is open in S^2, since $N = H \cap S^2$, where H is the open set in $\mathbb{R}^3$ given by

$$H = \{(x, y, z) \in \mathbb{R}^3 : z > 0\}.$$

Note that N is not itself an open set in $\mathbb{R}^3$. Indeed the point $(0, 0, 1)$ belongs to N, but, for any positive real number δ , the open ball (in $\mathbb{R}^3$) of radius δ centred on $(0, 0, 1)$ contains points (x, y, z) for which $x^2 + y^2 + z^2 \neq 1$. Thus the open ball of radius δ centred on the point $(0, 0, 1)$ is not a subset of N.

Lemma 4.1

Let X be a subset of $\mathbb{R}^n$, and let $\mathbf{p}$ be a point of X. Then, for any positive real number η, the open ball $B_X(\mathbf{p}, \eta)$ in X of radius η centred on $\mathbf{p}$ is open in X.

Proof

Let $\mathbf{q}$ be an element of $B_X(\mathbf{p}, \eta)$. We must show that there exists some positive real number δ such that $B_X(\mathbf{q}, \delta) \subset B_X(\mathbf{p}, \eta)$. Let $\delta = \eta - |\mathbf{q} - \mathbf{p}|$. Then $\delta > 0$, since $|\mathbf{q} - \mathbf{p}| < \eta$. Moreover if $\mathbf{x} \in B_X(\mathbf{q}, \delta)$ then

$$|\mathbf{x} - \mathbf{p}| \leq |\mathbf{x} - \mathbf{q}| + |\mathbf{q} - \mathbf{p}| < \delta + |\mathbf{q} - \mathbf{p}| = \eta,$$

by the Triangle Inequality, and hence $\mathbf{x} \in B_X(\mathbf{p}, \eta)$. Thus $B_X(\mathbf{q}, \delta) \subset B_X(\mathbf{p}, \eta)$. This shows that $B_X(\mathbf{p}, \eta)$ is an open set, as required. ∎

Lemma 4.2

Let X be a subset of $\mathbb{R}^n$, and let $\mathbf{p}$ be a point of X. Then, for any non-negative real number η, the set $\{\mathbf{x} \in X : |\mathbf{x} - \mathbf{p}| > \eta\}$ is an open set in X.

Proof

Let $\mathbf{q}$ be a point of X satisfying $|\mathbf{q} - \mathbf{p}| > \eta$, and let $\mathbf{x}$ be any point of X satisfying $|\mathbf{x} - \mathbf{q}| < \delta$, where $\delta = |\mathbf{q} - \mathbf{p}| - \eta$. Then

$$|\mathbf{q} - \mathbf{p}| \leq |\mathbf{q} - \mathbf{x}| + |\mathbf{x} - \mathbf{p}|,$$

by the Triangle Inequality. It follows that

$$|\mathbf{x} - \mathbf{p}| \geq |\mathbf{q} - \mathbf{p}| - |\mathbf{x} - \mathbf{q}| > |\mathbf{q} - \mathbf{p}| - \delta = \eta.$$

Thus $B_X(\mathbf{q}, \delta)$ is contained in the given set. The result follows. ∎

4.3. Convergence of Sequences and Open Sets

Lemma 4.3

An infinite sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ of points in $\mathbb{R}^n$ converges to a point $\mathbf{p}$ if and only if, given any open set V which contains $\mathbf{p}$, there exists some positive integer N such that $\mathbf{x}_j \in V$ for all positive integers j satisfying $j \geq N$.

Proof

Suppose that the infinite sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ of points in $\mathbb{R}^n$ has the property that, given any open set V which contains $\mathbf{p}$, there exists some positive integer N such that $\mathbf{x}_j \in V$ whenever $j \geq N$. Let some positive real number ε be given. The open ball $B(\mathbf{p}, \varepsilon)$ of radius ε centred on the point $\mathbf{p}$ is an open set by Lemma 4.1. Therefore there exists some positive integer N such that $\mathbf{x}_j \in B(\mathbf{p}, \varepsilon)$ whenever $j \geq N$. Thus $|\mathbf{x}_j - \mathbf{p}| < \varepsilon$ whenever $j \geq N$. This shows that the infinite sequence converges to the point $\mathbf{p}$.

Conversely, suppose that the infinite sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ of points of $\mathbb{R}^n$ converges to the point $\mathbf{p}$. Let V be an open set to which that point $\mathbf{p}$ belongs. Then there exists some positive real number ε such that the open ball $B(\mathbf{p}, \varepsilon)$ of radius ε centred on $\mathbf{p}$ is a subset of V. All points $\mathbf{x}$ of $\mathbb{R}^n$ that satisfy $|\mathbf{x} - \mathbf{p}| < \varepsilon$ then belong to the open set V. But there exists some positive integer N with the property that $|\mathbf{x}_j - \mathbf{p}| < \varepsilon$ whenever $j \geq N$, since the sequence converges to $\mathbf{p}$. Therefore $\mathbf{x}_j \in V$ whenever $j \geq N$, as required. $\blacksquare$

4.4. The Topology of Euclidean Spaces

Proposition 4.4

Let X be a subset of $\mathbb{R}^n$. The collection of open sets in X has the following properties:—

- **(i)** *the empty set $\emptyset$ and the whole set X are both open in X;*
- **(ii)** *the union of any collection of open sets in X is itself open in X;*
- **(iii)** *the intersection of any finite collection of open sets in X is itself open in X.*

Proof

The empty set $\emptyset$ is an open set by convention. Moreover the definition of an open set is satisfied trivially by the whole set X. This proves (i).

Let $\mathcal{C}$ be any collection of open sets in X, and let W denote the union of all the open sets belonging to $\mathcal{C}$. We must show that W is itself open in X. Let $\mathbf{p} \in W$. Then $\mathbf{p} \in V$ for some set V belonging to the collection $\mathcal{C}$. It follows that there exists some positive real number δ such that $B_X(\mathbf{p}, \delta) \subset V$. But $V \subset W$, and thus $B_X(\mathbf{p}, \delta) \subset W$. This shows that W is open in X. This proves (ii).

Finally let $V_1, V_2, V_3, \ldots, V_k$ be a *finite* collection of subsets of X that are open in X, and let V denote the intersection $V_1 \cap V_2 \cap \cdots \cap V_k$ of these sets. Let $\mathbf{p} \in V$. Now $\mathbf{p} \in V_j$ for $j = 1, 2, \ldots, k$, and therefore there exist strictly positive real numbers $\delta_1, \delta_2, \ldots, \delta_k$ such that $B_X(\mathbf{p}, \delta_j) \subset V_j$ for $j = 1, 2, \ldots, k$. Let δ be the minimum of $\delta_1, \delta_2, \ldots, \delta_k$. Then $\delta > 0$. (This is where we need the fact that we are dealing with a finite collection of sets.) Now $B_X(\mathbf{p}, \delta) \subset B_X(\mathbf{p}, \delta_j) \subset V_j$ for $j = 1, 2, \ldots, k$, and thus $B_X(\mathbf{p}, \delta) \subset V$. Thus the intersection V of the sets $V_1, V_2, \ldots, V_k$ is itself open in X. This proves (iii). ∎

Example

The set $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 < 4 \text{ and } z > 1\}$ is an open set in $\mathbb{R}^3$, since it is the intersection of the open ball of radius 2 centred on the origin with the open set $\{(x, y, z) \in \mathbb{R}^3 : z > 1\}$.

Example

The set $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 < 4 \text{ or } z > 1\}$ is an open set in $\mathbb{R}^3$, since it is the union of the open ball of radius 2 centred on the origin with the open set $\{(x, y, z) \in \mathbb{R}^3 : z > 1\}$.

Example

The set

$$\{(x, y, z) \in \mathbb{R}^3 : (x - n)^2 + y^2 + z^2 < \tfrac{1}{4} \text{ for some } n \in \mathbb{Z}\}$$

is an open set in $\mathbb{R}^3$, since it is the union of the open balls of radius $\frac{1}{2}$ centred on the points $(n, 0, 0)$ for all integers n.

Example

For each positive integer k, let

$$V_k = \{(x, y, z) \in \mathbb{R}^3 : k^2(x^2 + y^2 + z^2) < 1\}.$$

Now each set V_k is an open ball of radius $1/k$ centred on the origin, and is therefore an open set in $\mathbb{R}^3$. However the intersection of the sets V_k for all positive integers k is the set $\{(0, 0, 0)\}$, and thus the intersection of the sets V_k for all positive integers k is not itself an open set in $\mathbb{R}^3$. This example demonstrates that infinite intersections of open sets need not be open.

Proposition 4.5

Let X be a subset of $\mathbb{R}^n$, and let W be a subset of X. Then W is open in X if and only if there exists some open set V in $\mathbb{R}^n$ for which $W = V \cap X$.

Proof

First suppose that $W = V \cap X$ for some open set V in $\mathbb{R}^n$. Let $\mathbf{p} \in W$. Then the definition of open sets in $\mathbb{R}^n$ ensures that there exists some positive real number δ such that

$$\{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x} - \mathbf{p}| < \delta\} \subset V.$$

Then

$$\{\mathbf{x} \in X : |\mathbf{x} - \mathbf{p}| < \delta\} \subset W.$$

This shows that W is open in X.

Conversely suppose that the subset W of X is open in X. For each point $\mathbf{p}$ of W there exists some positive real number $\delta_{\mathbf{p}}$ such that

$$\{\mathbf{x} \in X : |\mathbf{x} - \mathbf{p}| < \delta_{\mathbf{p}}\} \subset W.$$

For each $\mathbf{p} \in W$, let $B(\mathbf{p}, \delta_{\mathbf{p}})$ denote the open ball in $\mathbb{R}^n$ of radius $\delta_{\mathbf{p}}$ centred on the point $\mathbf{p}$, so that

$$B(\mathbf{p}, \delta_{\mathbf{p}}) = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x} - \mathbf{p}| < \delta_{\mathbf{p}}\}$$

for all $\mathbf{p} \in W$, and let V be the union of all the open balls $B(\mathbf{p}, \delta_{\mathbf{p}})$ as $\mathbf{p}$ ranges over all the points of W. Then V is an open set in $\mathbb{R}^n$.

Indeed every open ball in $\mathbb{R}^n$ is an open set (Lemma 4.1), and any union of open sets in $\mathbb{R}^n$ is itself an open set (Proposition 4.4). The set V is a union of open balls. It is therefore a union of open sets, and so must itself be an open set.

Now $B(\mathbf{p}, \delta_{\mathbf{p}}) \cap X \subset W$. for all $\mathbf{p} \in W$. Also every point of V belongs to $B(\mathbf{p}, \delta_{\mathbf{p}})$ for at least one point $\mathbf{p}$ of W. It follows that $V \cap X \subset W$. But $\mathbf{p} \in B(\mathbf{p}, \delta_{\mathbf{p}})$ and $B(\mathbf{p}, \delta_{\mathbf{p}}) \subset V$ for all $\mathbf{p} \in W$, and therefore $W \subset V$, and thus $W \subset V \cap X$. It follows that $W = V \cap X$, as required. ∎

4.5. Closed Sets in Euclidean Spaces

Definition

Let X be a subset of $\mathbb{R}^n$. A subset F of X is said to be *closed* in X if and only if its complement $X \setminus F$ in X is open in X.

(Recall that $X \setminus F = \{\mathbf{x} \in X : \mathbf{x} \notin F\}$.)

Example

The sets $\{(x, y, z) \in \mathbb{R}^3 : z \geq c\}$, $\{(x, y, z) \in \mathbb{R}^3 : z \leq c\}$, and $\{(x, y, z) \in \mathbb{R}^3 : z = c\}$ are closed sets in $\mathbb{R}^3$ for each real number c, since the complements of these sets are open in $\mathbb{R}^3$.

Example

Let X be a subset of $\mathbb{R}^n$, let $\mathbf{p}$ be a point of X, and let η be a non-negative real number. Then the sets $\{\mathbf{x} \in X : |\mathbf{x} - \mathbf{p}| \leq \eta\}$ and $\{\mathbf{x} \in X : |\mathbf{x} - \mathbf{p}| \geq \eta\}$ are closed in X. In particular, the set $\{\mathbf{p}\}$ consisting of the single point $\mathbf{p}$ is a closed set in X. (These results follow immediately using Lemma 4.1 and Lemma 4.2 and the definition of closed sets.)

Let $\mathcal{A}$ be some collection of subsets of a set X. Then

$$X \setminus \bigcup_{S \in \mathcal{A}} S = \bigcap_{S \in \mathcal{A}} (X \setminus S), \qquad X \setminus \bigcap_{S \in \mathcal{A}} S = \bigcup_{S \in \mathcal{A}} (X \setminus S)$$

(i.e., the complement of the union of some collection of subsets of X is the intersection of the complements of those sets, and the complement of the intersection of some collection of subsets of X is the union of the complements of those sets).

Indeed let $\mathcal{A}$ be some collection of subsets of a set X, and let $\mathbf{x}$ be a point of X. Then

$$\begin{aligned} \mathbf{x} \in X \setminus \bigcup_{S \in \mathcal{A}} S &\iff \mathbf{x} \notin \bigcup_{S \in \mathcal{A}} S \\ &\iff \text{for all } S \in \mathcal{A},\ \mathbf{x} \notin S \\ &\iff \text{for all } S \in \mathcal{A},\ \mathbf{x} \in X \setminus S \\ &\iff \mathbf{x} \in \bigcap_{S \in \mathcal{A}} (X \setminus S), \end{aligned}$$

and therefore

$$X \setminus \bigcup_{S \in \mathcal{A}} S = \bigcap_{S \in \mathcal{A}} (X \setminus S).$$

Again let $\mathbf{x}$ be a point of X. Then

$$\begin{aligned} \mathbf{x} \in X \setminus \bigcap_{S \in \mathcal{A}} S &\iff \mathbf{x} \notin \bigcap_{S \in \mathcal{A}} S \\ &\iff \text{there exists } S \in \mathcal{A} \text{ for which } \mathbf{x} \notin S \\ &\iff \text{there exists } S \in \mathcal{A} \text{ for which } \mathbf{x} \in X \setminus S \\ &\iff \mathbf{x} \in \bigcup_{S \in \mathcal{A}} (X \setminus S), \end{aligned}$$

and therefore

$$X \setminus \bigcap_{S \in \mathcal{A}} S = \bigcup_{S \in \mathcal{A}} (X \setminus S).$$

The following result therefore follows directly from Proposition 4.4.

Proposition 4.6

Let X be a subset of $\mathbb{R}^n$. The collection of closed sets in X has the following properties:—

(i) *the empty set $\emptyset$ and the whole set X are both closed in X;*

(ii) *the intersection of any collection of closed sets in X is itself closed in X;*

(iii) *the union of any finite collection of closed sets in X is itself closed in X.*

Proof

The empty set $\emptyset$ is the complement in X of the whole set X. The set X is open in itself. It follows that the empty set $\emptyset$ is closed in X.

The whole set X is the complement in X of the empty set. The empty set is open in X. It follows that the whole set X is closed in itself.

Next let $\mathcal{C}$ be a collection of subsets of X that are closed in X, and let G be the intersection of all the sets that are members of the collection $\mathcal{C}$. Now the complement in X of the set G, being the complement of the intersection of all the members of the collection $\mathcal{C}$ is the union of the complements of the members of this collection $\mathcal{C}$. Now the complement of each member of the collection $\mathcal{C}$ is open in X. Consequently the union of the complements of the members of the collection must also be open in X. Thus the complement of the set G is open in X, and therefore the set G itself is closed in X.

Now suppose that the collection $\mathcal{C}$ is a finite collection of subsets of X that are closed in X, and let H be the union of all the sets that are members of the finite collection $\mathcal{C}$. Now the complement in X of the set H, being the complement of the union of all the members of the finite collection $\mathcal{C}$ is the intersection of the complements of the members of this finite collection $\mathcal{C}$. Now the complement of each member of the finite collection $\mathcal{C}$ is open in X. Consequently the intersection of the complements of the members of the finite collection must also be open in X. Thus the complement of the set H is open in X, and therefore the set H itself is closed in X. This completes the proof. ∎

Lemma 4.7

Let X be a subset of $\mathbb{R}^n$, and let F be a subset of X which is closed in X. Let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ be an infinite sequence of points of F which converges to some point $\mathbf{p}$ of X. Then $\mathbf{p} \in F$.

Proof

The complement $X \setminus F$ of F in X is open, since F is closed. Suppose that $\mathbf{p}$ were a point belonging to $X \setminus F$. It would then follow from Lemma 4.3 that $\mathbf{x}_j \in X \setminus F$ for all values of j greater than some positive integer N, contradicting the fact that $\mathbf{x}_j \in F$ for all j. This contradiction shows that $\mathbf{p}$ must belong to F, as required. ∎