MAU23203—Analysis in Several Variables School of Mathematics, Trinity College Michaelmas Term 2021 Section 6: Limits of Functions of Several Real Variables

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# 6. Limits of Functions of Several Real Variables

6.1. Limit Points of Subsets of Euclidean Spaces

#### Definition

Let X be a subset of *n*-dimensional Euclidean space  $\mathbb{R}^n$ , and let  $\mathbf{p} \in \mathbb{R}^n$ . The point  $\mathbf{p}$  is said to be a *limit point* of the set X if, given any positive real number  $\delta$ , there exists some point  $\mathbf{x}$  of X for which  $0 < |\mathbf{x} - \mathbf{p}| < \delta$ .

# 6.2. Basic Properties of Limits of Functions of Several Real Variables

# Definition

Let X be a subset of *m*-dimensional Euclidean space  $\mathbb{R}^m$ , let  $\varphi: X \to \mathbb{R}^n$  be a function mapping the set X into *n*-dimensional Euclidean space  $\mathbb{R}^n$ , let **p** be a limit point of the set X, and let **v** be a vector in  $\mathbb{R}^n$ . The point **v** is said to be the *limit* of  $\varphi(\mathbf{x})$ , as **x** tends to **p** in X, if and only if, given any strictly positive real number  $\varepsilon$ , there exists some strictly positive real number  $\delta$  such that  $|\varphi(\mathbf{x}) - \mathbf{v}| < \varepsilon$  whenever  $\mathbf{x} \in X$  satisfies  $0 < |\mathbf{x} - \mathbf{p}| < \delta$ .

Let X be a subset of *m*-dimensional Euclidean space  $\mathbb{R}^m$ , let  $\varphi: X \to \mathbb{R}^n$  be a function mapping the set X into *n*-dimensional Euclidean space  $\mathbb{R}^n$ , let **p** be a limit point of the set X, and let **v** be a vector in  $\mathbb{R}^n$ . If **v** is the limit of  $\varphi(\mathbf{x})$  as **x** tends to **p** in X then we can denote this fact by writing  $\lim_{\mathbf{x}\to\mathbf{p}}\varphi(\mathbf{x}) = \mathbf{v}$ .

#### **Proposition 6.1**

Let X be a subset of  $\mathbb{R}^m$ , let **p** be a limit point of X, and let **v** be a vector in  $\mathbb{R}^n$ . A function  $\varphi \colon X \to \mathbb{R}^n$  has the property that

$$\lim_{\mathbf{x} \to \mathbf{p}} \varphi(\mathbf{x}) = \mathbf{v}$$

if and only if

$$\lim_{\mathbf{x}\to\mathbf{p}}f_i(\mathbf{x})=v_i$$

for i = 1, 2, ..., n, where  $f_1, f_2, ..., f_n$  are the components of the function  $\varphi$  and  $\mathbf{v} = (v_1, v_2, ..., v_n)$ .

Suppose that  $\lim_{\mathbf{x}\to\mathbf{p}}\varphi(\mathbf{x}) = \mathbf{v}$ . Let i be an integer between 1 and n, and let some positive real number  $\varepsilon$  be given. Then there exists some positive real number  $\delta$  such that  $|\varphi(\mathbf{x}) - \mathbf{v}| < \varepsilon$  whenever  $0 < |\mathbf{x} - \mathbf{p}| < \delta$ . It then follows from the definition of the Euclidean norm that

$$|f_i(\mathbf{x}) - \mathbf{v}_i| \leq |arphi(\mathbf{x}) - \mathbf{v}| < arepsilon$$

whenever  $0 < |\mathbf{x} - \mathbf{p}| < \delta$ . Thus if  $\lim_{\mathbf{x} \to \mathbf{p}} \varphi(\mathbf{x}) = \mathbf{v}$  then  $\lim_{\mathbf{x} \to \mathbf{p}} f_i(\mathbf{x}) = v_i$  for i = 1, 2, ..., n. Conversely suppose that

$$\lim_{\mathbf{x}\to\mathbf{p}}f_i(\mathbf{x})=v_i$$

for i = 1, 2, ..., n. Let some positive real number  $\varepsilon$  be given. Then there exist positive real numbers  $\delta_1, \delta_2, ..., \delta_n$  such that  $|f_i(\mathbf{x}) - v_i| < \varepsilon / \sqrt{n}$  for  $\mathbf{x} \in X$  satisfying  $0 < |\mathbf{x} - \mathbf{p}| < \delta_i$ . Let  $\delta$  be the minimum of  $\delta_1, \delta_2, ..., \delta_n$ . If  $\mathbf{x} \in X$  satisfies  $0 < |\mathbf{x} - \mathbf{p}| < \delta$ then

$$|\varphi(\mathbf{x}) - \mathbf{v}|^2 = \sum_{i=1}^n (f_i(\mathbf{x}) - v_i)^2 < \varepsilon^2,$$

and hence  $|\varphi(\mathbf{x}) - \mathbf{v}| < \varepsilon$ . Thus

$$\lim_{\mathbf{x}\to\mathbf{p}}\varphi(\mathbf{x})=\mathbf{v},$$

# **Proposition 6.2**

Let X be a subset of m-dimensional Euclidean space  $\mathbb{R}^m$ , let  $\varphi: X \to \mathbb{R}^n$  and  $\psi: X \to \mathbb{R}^n$  be functions mapping X into n-dimensional Euclidean space  $\mathbb{R}^n$ , let **p** be a limit point of X, and let **v** and **w** be points of  $\mathbb{R}^n$ . Suppose that

$$\lim_{\mathsf{x} o \mathsf{p}} arphi(\mathsf{x}) = \mathsf{v}$$

and

$$\lim_{\mathbf{x}\to\mathbf{p}}\psi(\mathbf{x})=\mathbf{w}.$$

Then

$$\lim_{\mathbf{x}\to\mathbf{p}}(\varphi(\mathbf{x})+\psi(\mathbf{x}))=\mathbf{v}+\mathbf{w}.$$

Let some strictly positive real number  $\varepsilon$  be given. Then there exist strictly positive real numbers  $\delta_1$  and  $\delta_2$  such that

$$|\varphi(\mathbf{x}) - \mathbf{v}| < \frac{1}{2}\varepsilon$$

whenever  $\mathbf{x} \in X$  satisfies  $0 < |\mathbf{x} - \mathbf{p}| < \delta_1$  and

$$|\psi(\mathbf{x}) - \mathbf{w}| < \frac{1}{2}\varepsilon$$

whenever  $\mathbf{x} \in X$  satisfies  $0 < |\mathbf{x} - \mathbf{p}| < \delta_2$ .

# 6. Limits of Functions of Several Real Variables (continued)

Let  $\delta$  be the minimum of  $\delta_1$  and  $\delta_2$ . Then  $\delta > 0$ , and if  $\mathbf{x} \in X$  satisfies  $0 < |\mathbf{x} - \mathbf{p}| < \delta$  then

$$|arphi(\mathbf{x}) - \mathbf{v}| < rac{1}{2}arepsilon$$

and

$$|\psi(\mathbf{x}) - \mathbf{w}| < rac{1}{2}arepsilon,$$

and therefore

$$\begin{aligned} |\varphi(\mathbf{x}) + \psi(\mathbf{x}) - (\mathbf{v} + \mathbf{w})| &\leq |\varphi(\mathbf{x}) - \mathbf{v}| + |\psi(\mathbf{x}) - \mathbf{w}| \\ &< \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon. \end{aligned}$$

It follows that

$$\lim_{\mathbf{x}\to\mathbf{p}}(\varphi(\mathbf{x})+\psi(\mathbf{x}))=\mathbf{v}+\mathbf{w},$$

#### Lemma 6.3

Let X and Y be subsets of  $\mathbb{R}^m$  and  $\mathbb{R}^n$  respectively, let **p** be a limit point of X, let **v** be a point of Y, let  $\varphi \colon X \to Y$  be a function mapping the set X into the set Y, and let  $\psi \colon Y \to \mathbb{R}^k$  be a function mapping the set Y into  $\mathbb{R}^k$ . Suppose that

 $\lim_{\mathbf{x} \to \mathbf{p}} \varphi(\mathbf{x}) = \mathbf{v}$ 

and that the function  $\psi$  is continuous at **v**. Then

 $\lim_{\mathbf{x}\to\mathbf{p}}\psi(\varphi(\mathbf{x}))=\psi(\mathbf{v}).$ 

Let some positive real number  $\varepsilon$  be given. Then there exists some positive real number  $\eta$  such that  $|\psi(\mathbf{y}) - \psi(\mathbf{v})| < \varepsilon$  for all  $\mathbf{y} \in \mathbf{Y}$  satisfying  $|\mathbf{y} - \mathbf{v}| < \eta$ , because the function g is continuous at  $\mathbf{v}$ . But then there exists some positive real number  $\delta$  such that  $|\varphi(\mathbf{x}) - \mathbf{v}| < \eta$  for all  $\mathbf{x} \in X$  satisfying  $0 < |\mathbf{x} - \mathbf{p}| < \delta$ . It follows that  $|\psi(\varphi(\mathbf{x})) - \psi(\mathbf{v})| < \varepsilon$  for all  $\mathbf{x} \in X$  satisfying  $0 < |\mathbf{x} - \mathbf{p}| < \delta$ . It follows that  $|\psi(\varphi(\mathbf{x})) - \psi(\mathbf{v})| < \varepsilon$  for all  $\mathbf{x} \in X$  satisfying  $0 < |\mathbf{x} - \mathbf{p}| < \delta$ , and thus

$$\lim_{\mathbf{x}\to\mathbf{p}}\psi(\varphi(\mathbf{x}))=\psi(\mathbf{v}),$$

### **Proposition 6.4**

Let X be a subset of  $\mathbb{R}^m$ , let  $f: X \to \mathbb{R}$  and  $g: X \to \mathbb{R}$  be real-valued functions on X, and let **p** be a limit point of the set X. Suppose that  $\lim_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x})$  and  $\lim_{\mathbf{x}\to\mathbf{p}} g(\mathbf{x})$  both exist. Then so do  $\lim_{\mathbf{x}\to\mathbf{p}} (f(\mathbf{x}) + g(\mathbf{x})), \lim_{\mathbf{x}\to\mathbf{p}} (f(\mathbf{x}) - g(\mathbf{x}))$  and  $\lim_{\mathbf{x}\to\mathbf{p}} (f(\mathbf{x})g(\mathbf{x}))$ , and moreover

$$\begin{split} &\lim_{\mathbf{x}\to\mathbf{p}}(f(\mathbf{x})+g(\mathbf{x})) &= \lim_{\mathbf{x}\to\mathbf{p}}f(\mathbf{x})+\lim_{\mathbf{x}\to\mathbf{p}}g(\mathbf{x}),\\ &\lim_{\mathbf{x}\to\mathbf{p}}(f(\mathbf{x})-g(\mathbf{x})) &= \lim_{\mathbf{x}\to\mathbf{p}}f(\mathbf{x})-\lim_{\mathbf{x}\to\mathbf{p}}g(\mathbf{x}),\\ &\lim_{\mathbf{x}\to\mathbf{p}}(f(\mathbf{x})g(\mathbf{x})) &= \lim_{\mathbf{x}\to\mathbf{p}}f(\mathbf{x})\times\lim_{\mathbf{x}\to\mathbf{p}}g(\mathbf{x}), \end{split}$$

If moreover 
$$g(\mathbf{x}) \neq 0$$
 for all  $\mathbf{x} \in X$  and  $\lim_{\mathbf{x} \to \mathbf{p}} g(\mathbf{x}) \neq 0$  then  
$$\lim_{\mathbf{x} \to \mathbf{p}} \frac{f(\mathbf{x})}{g(\mathbf{x})} = \frac{\lim_{\mathbf{x} \to \mathbf{p}} f(\mathbf{x})}{\lim_{\mathbf{x} \to \mathbf{p}} g(\mathbf{x})}.$$

Let  $q = \lim_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x})$  and  $r = \lim_{\mathbf{x}\to\mathbf{p}} g(\mathbf{x})$ , and let  $\psi \colon X \to \mathbb{R}^2$  be defined such that

$$\psi(\mathbf{x}) = (f(\mathbf{x}), g(\mathbf{x}))$$

for all  $\mathbf{x} \in X$ . Then

$$\lim_{\mathbf{x}\to\mathbf{p}}\psi(\mathbf{x})=(q,r)$$

(see Proposition 6.1).

#### 6. Limits of Functions of Several Real Variables (continued)

Let  $s: \mathbb{R}^2 \to \mathbb{R}$  and  $m: \mathbb{R}^2 \to \mathbb{R}$  be the functions from  $\mathbb{R}^2$  to  $\mathbb{R}$  defined such that s(u, v) = u + v and m(u, v) = uv for all  $u, v \in \mathbb{R}$ . Then the functions s and m are continuous (see Lemma 5.4). Also  $f + g = s \circ \psi$  and  $f \cdot g = m \circ \psi$ . It follows from this that

$$\begin{split} \lim_{\mathbf{x}\to\mathbf{p}}(f(\mathbf{x})+g(\mathbf{x})) &= \lim_{\mathbf{x}\to\mathbf{p}}s(f(\mathbf{x}),g(\mathbf{x})) = \lim_{\mathbf{x}\to\mathbf{p}}s(\psi(\mathbf{x}))\\ &= s\left(\lim_{\mathbf{x}\to\mathbf{p}}\psi(\mathbf{x})\right) = s(q,r) = q+r, \end{split}$$

(see Lemma 6.3), and

$$\lim_{\mathbf{x}\to\mathbf{p}}(-g(\mathbf{x}))=-r.$$

It follows that

$$\lim_{\mathbf{x}\to\mathbf{p}}(f(\mathbf{x})-g(\mathbf{x}))=q-r.$$

Similarly, when taking limits of products of functions,

$$\lim_{\mathbf{x}\to\mathbf{p}}(f(\mathbf{x})g(\mathbf{x})) = \lim_{\mathbf{x}\to\mathbf{p}} m(f(\mathbf{x}),g(\mathbf{x})) = \lim_{\mathbf{x}\to\mathbf{p}} m(\psi(\mathbf{x}))$$
$$= m\left(\lim_{\mathbf{x}\to\mathbf{p}}\psi(\mathbf{x})\right) = m(q,r) = qr$$

Now suppose that  $g(\mathbf{x}) \neq 0$  for all  $\mathbf{x} \in X$  and that  $\lim_{\mathbf{x} \to \mathbf{p}} g(\mathbf{x}) \neq 0$ . Representing the function sending  $\mathbf{x} \in X$  to  $1/g(\mathbf{x})$  as the composition of the function g and the reciprocal function  $e \colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$ , where e(t) = 1/t for all non-zero real numbers t, we find, as in the first proof, that the function sending each point  $\mathbf{x}$  of X to

$$\lim_{\mathbf{x}\to\mathbf{p}}\left(\frac{1}{g(\mathbf{x})}\right)=\frac{1}{r}$$

It then follows that

$$\lim_{\mathbf{x}\to\mathbf{p}}\frac{f(\mathbf{x})}{g(\mathbf{x})}=\frac{q}{r},$$

# 6.3. Relationships between Limits and Continuity

# **Proposition 6.5**

Let X be a subset of  $\mathbb{R}^m$ , let  $f: X \to \mathbb{R}^n$  be a function mapping the set X into  $\mathbb{R}^n$ , and let **p** be a point of the set X that is also a limit point of X. Then the function f is continuous at the point **p** if and only if  $\lim_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x}) = f(\mathbf{p})$ .

# Proof

The result follows directly on comparing the relevant definitions.

Let X be a subset of *m*-dimensional Euclidean space  $\mathbb{R}^m$ , and let **p** be a point of the set X. Suppose that the point **p** is not a limit point of the set X. Then there exists some strictly positive real number  $\delta_0$  such that  $|\mathbf{x} - \mathbf{p}| \ge \delta_0$  for all  $\mathbf{x} \in X$  satisfying  $\mathbf{x} \neq \mathbf{p}$ . The point **p** is then said to be an *isolated point* of X.

Let X be a subset of *m*-dimensional Euclidean space  $\mathbb{R}^m$ . The definition of continuity then ensures that any function  $\varphi \colon X \to \mathbb{R}^n$  mapping the set X into *n*-dimensional Euclidean space  $\mathbb{R}^n$  is continuous at any isolated point of its domain X.