MAU23203—Analysis in Several Variables
School of Mathematics, Trinity College
Michaelmas Term 2021
Section 5: Continuous Functions of Several Real Variables

Trinity College Dublin

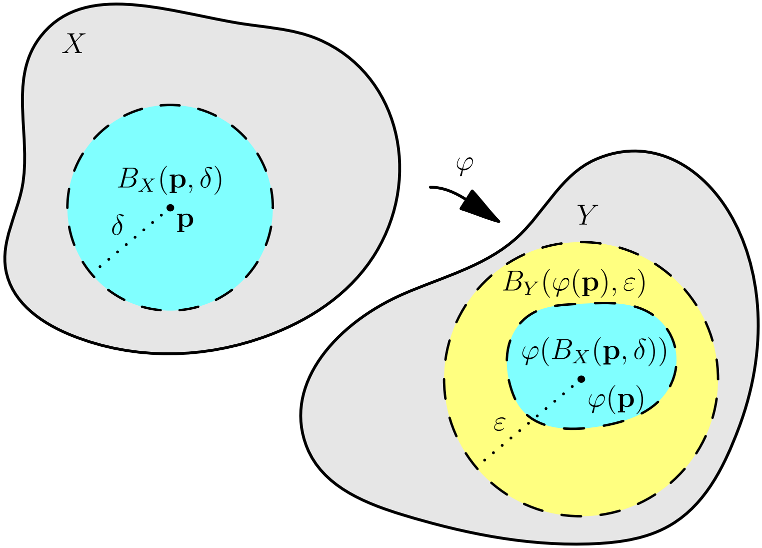
5. Continuous Functions of Several Real Variables

5.1. The Concept and Basic Properties of Continuity

Definition

Let X and Y be subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively. A function $\varphi\colon X \to Y$ from X to Y is said to be *continuous* at a point $\mathbf{p}$ of X if and only if, given any strictly positive real number ε, there exists some strictly positive real number δ such that $|\varphi(\mathbf{x}) - \varphi(\mathbf{p})| < \varepsilon$ whenever $\mathbf{x} \in X$ satisfies $|\mathbf{x} - \mathbf{p}| < \delta$.

The function $\varphi\colon X \to Y$ is said to be continuous on X if and only if it is continuous at every point $\mathbf{p}$ of X.



X
$B_X(\mathbf{p}, \delta)$
δ
$\mathbf{p}$
φ
Y
$B_Y(\varphi(\mathbf{p}), \varepsilon)$
$\varphi(B_X(\mathbf{p}, \delta))$
$\varphi(\mathbf{p})$
ε

Proposition 5.1

Let X, Y and Z be subsets of Euclidean spaces, let $\varphi\colon X \to Y$ be a function from X to Y and let $\psi\colon Y \to Z$ be a function from Y to Z. Suppose that φ is continuous at some point $\mathbf{p}$ of X and that ψ is continuous at $\varphi(\mathbf{p})$. Then the composition function $\psi \circ \varphi\colon X \to Z$ is continuous at $\mathbf{p}$.

Proof

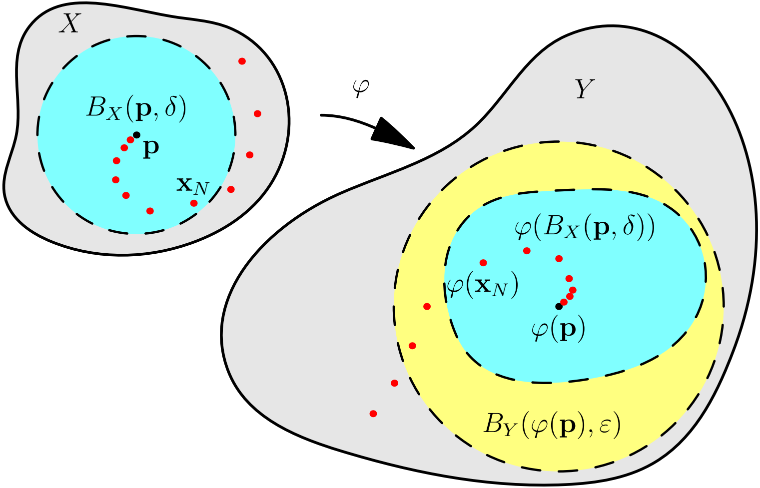
Let $\mathbf{q} = \varphi(\mathbf{p})$, and let some positive real number ε be given. Then there exists some positive real number η such that $|\psi(\mathbf{y}) - \psi(\mathbf{q})| < \varepsilon$ for all $\mathbf{y} \in Y$ satisfying $|\mathbf{y} - \mathbf{q}| < \eta$. But then there exists some positive real number δ such that $|\varphi(\mathbf{x}) - \mathbf{q}| < \eta$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$. It follows that $|\psi(\varphi(\mathbf{x})) - \psi(\varphi(\mathbf{p}))| < \varepsilon$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$, and thus $\psi \circ \varphi$ is continuous at $\mathbf{p}$, as required. $\blacksquare$

Proposition 5.2

Let X and Y be subsets of Euclidean spaces, and let $\varphi\colon X \to Y$ be a continuous function from X to Y. Let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ be an infinite sequence of points of X which converges to some point $\mathbf{p}$ of X. Then the sequence $\varphi(\mathbf{x}_1), \varphi(\mathbf{x}_2), \varphi(\mathbf{x}_3), \ldots$ converges to $\varphi(\mathbf{p})$.

Proof

Let some positive real number ε be given. The function φ is continuous at $\mathbf{p}$, and therefore there exists some positive real number δ such that $|\varphi(\mathbf{x}) - \varphi(\mathbf{p})| < \varepsilon$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$. Also the infinite sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ converges to the point $\mathbf{p}$, and therefore there exists some positive integer N such that $|\mathbf{x}_j - \mathbf{p}| < \delta$ whenever $j \geq N$. It follows that if $j \geq N$ then $|\varphi(\mathbf{x}_j) - \varphi(\mathbf{p})| < \varepsilon$. Thus the sequence $\varphi(\mathbf{x}_1), \varphi(\mathbf{x}_2), \varphi(\mathbf{x}_3), \ldots$ converges to $\varphi(\mathbf{p})$, as required. ∎



X
$B_X(\mathbf{p}, \delta)$
$\mathbf{p}$
$\mathbf{x}_N$
φ
Y
$\varphi(B_X(\mathbf{p}, \delta))$
$\varphi(\mathbf{x}_N)$
$\varphi(\mathbf{p})$
$B_Y(\varphi(\mathbf{p}), \varepsilon)$

Let X and Y be subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively, and let $\varphi\colon X \to Y$ be a function from X to Y. Then

$$\varphi(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_n(\mathbf{x}))$$

for all $\mathbf{x} \in X$, where $f_1, f_2, \ldots, f_n$ are functions from X to $\mathbb{R}$, referred to as the *components* of the function φ.

Proposition 5.3

Let X and Y be subsets of Euclidean spaces, and let $\mathbf{p} \in X$. A function $\varphi\colon X \to Y$ is continuous at the point $\mathbf{p}$ if and only if its components are all continuous at $\mathbf{p}$.

Proof

Let Y be a subset of n-dimensional Euclidean space $\mathbb{R}^n$. Note that the ith component f_i of φ is given by $f_i = \pi_i \circ f$, where $\pi_i\colon \mathbb{R}^n \to \mathbb{R}$ is the continuous function which maps $(y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$ onto its ith component y_i. Now any composition of continuous functions is continuous, by Proposition 5.1. Thus if φ is continuous at $\mathbf{p}$, then so are the components of φ.

Conversely suppose that the components of φ are continuous at $\mathbf{p} \in X$. Let some positive real number ε be given. Then there exist positive real numbers $\delta_1, \delta_2, \ldots, \delta_n$ such that $|f_i(\mathbf{x}) - f_i(\mathbf{p})| < \varepsilon/\sqrt{n}$ for $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta_i$. Let δ be the minimum of $\delta_1, \delta_2, \ldots, \delta_n$. If $\mathbf{x} \in X$ satisfies $|\mathbf{x} - \mathbf{p}| < \delta$ then

$$|\varphi(\mathbf{x}) - \varphi(\mathbf{p})|^2 = \sum_{i=1}^{n} |f_i(\mathbf{x}) - f_i(\mathbf{p})|^2 < \varepsilon^2,$$

and hence $|\varphi(\mathbf{x}) - \varphi(\mathbf{p})| < \varepsilon$. Thus the function φ is continuous at $\mathbf{p}$, as required. ∎

Lemma 5.4

Let functions $s\colon \mathbb{R}^2 \to \mathbb{R}$ and $m\colon \mathbb{R}^2 \to \mathbb{R}$ be defined so that $s(x, y) = x + y$ and $m(x, y) = xy$ for all real numbers x and y. Then the functions s and m are continuous.

Proof

Let $(u, v) \in \mathbb{R}^2$. We first show that $s\colon \mathbb{R}^2 \to \mathbb{R}$ is continuous at (u, v). Let some positive real number ε be given. Let $\delta = \frac{1}{2}\varepsilon$. If (x, y) is any point of $\mathbb{R}^2$ whose distance from (u, v) is less than δ then $|x - u| < \delta$ and $|y - v| < \delta$, and hence

$$|s(x, y) - s(u, v)| = |x + y - u - v| \leq |x - u| + |y - v| < 2\delta = \varepsilon.$$

This shows that $s\colon \mathbb{R}^2 \to \mathbb{R}$ is continuous at (u, v).

Next we show that $m\colon \mathbb{R}^2 \to \mathbb{R}$ is continuous at (u, v). Let some positive real number ε be given. Now

$$m(x, y) - m(u, v) = xy - uv = (x - u)(y - v) + u(y - v) + (x - u)v.$$

for all points (x, y) of $\mathbb{R}^2$. Thus if the distance from (x, y) to (u, v) is less than δ then $|x - u| < \delta$ and $|y - v| < \delta$, and hence $|m(x, y) - m(u, v)| < \delta^2 + (|u| + |v|)\delta$. Consequently if the positive real number δ is chosen to be the minimum of 1 and $\varepsilon/(1 + |u| + |v|)$ then $\delta^2 + (|u| + |v|)\delta \leq (1 + |u| + |v|)\delta \leq \varepsilon$, and thus $|m(x, y) - m(u, v)| < \varepsilon$ for all points (x, y) of $\mathbb{R}^2$ whose distance from (u, v) is less than δ. This shows that $m\colon \mathbb{R}^2 \to \mathbb{R}$ is continuous at (u, v). $\blacksquare$

Proposition 5.5

Let X be a subset of $\mathbb{R}^n$, and let $f\colon X \to \mathbb{R}$ and $g\colon X \to \mathbb{R}$ be continuous functions from X to $\mathbb{R}$. Then the functions $f+g$, $f-g$ and $f \cdot g$ are continuous. If in addition $g(\mathbf{x}) \neq 0$ for all $\mathbf{x} \in X$ then the quotient function f/g is continuous.

Proof

Note that $f+g = s \circ \psi$ and $f \cdot g = m \circ \psi$, where the functions $\psi\colon X \to \mathbb{R}^2$, $s\colon \mathbb{R}^2 \to \mathbb{R}$ and $m\colon \mathbb{R}^2 \to \mathbb{R}$ are defined so that $\psi(\mathbf{x}) = (f(\mathbf{x}), g(\mathbf{x}))$, $s(u,v) = u+v$ and $m(u,v) = uv$ for all $\mathbf{x} \in X$ and $u, v \in \mathbb{R}$. It follows from Proposition 5.3, Lemma 5.4 and Proposition 5.1 that $f+g$ and $f \cdot g$ are continuous, being compositions of continuous functions. Now $f - g = f + (-g)$, and both f and $-g$ are continuous. Therefore $f-g$ is continuous.

Now suppose that $g(\mathbf{x}) \neq 0$ for all $\mathbf{x} \in X$. Note that $1/g = r \circ g$, where $r \colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is the reciprocal function, defined so that $r(t) = 1/t$ for all non-zero real numbers t. Now the reciprocal function r is continuous. Thus the function $1/g$ is a composition of continuous functions and is thus continuous. But then, using the fact that a product of continuous real-valued functions is continuous, we deduce that f/g is continuous. $\blacksquare$

Example

Consider the function $\varphi\colon \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}^2$ defined so that

$$\varphi(x,y) = \left(\frac{x}{x^2+y^2}, \frac{-y}{x^2+y^2} \right)$$

for all real numbers x and y that are not both zero. The continuity of the components of this function φ follows from straightforward applications of Proposition 5.5. It then follows from Proposition 5.3 that the function φ is continuous on $\mathbb{R}^2 \setminus \{(0,0)\}$.

Lemma 5.6

Let X be a subset of $\mathbb{R}^m$, let $\varphi\colon X \to \mathbb{R}^n$ be a continuous function mapping X into $\mathbb{R}^n$, and let $|\varphi|\colon X \to \mathbb{R}$ be the real-valued function on X defined such that $|\varphi|(\mathbf{x}) = |\varphi(\mathbf{x})|$ for all $\mathbf{x} \in X$. Then the real-valued function $|\varphi|$ is continuous on X.

Proof

Let $\mathbf{x}$ and $\mathbf{p}$ be points of X. Then

$$|\varphi(\mathbf{x})| = |(\varphi(\mathbf{x}) - \varphi(\mathbf{p})) + \varphi(\mathbf{p})| \leq |\varphi(\mathbf{x}) - \varphi(\mathbf{p})| + |\varphi(\mathbf{p})|$$

and

$$|\varphi(\mathbf{p})| = |(\varphi(\mathbf{p}) - \varphi(\mathbf{x})) + \varphi(\mathbf{x})| \leq |\varphi(\mathbf{x}) - \varphi(\mathbf{p})| + |\varphi(\mathbf{x})|,$$

and therefore

$$\Big| |\varphi(\mathbf{x})| - |\varphi(\mathbf{p})| \Big| \leq |\varphi(\mathbf{x}) - \varphi(\mathbf{p})|.$$

The result now follows on applying the definition of continuity, using the above inequality. Indeed let $\mathbf{p}$ be a point of X, and let some positive real number ε be given. Then there exists a positive real number δ small enough to ensure that $|\varphi(\mathbf{x}) - \varphi(\mathbf{p})| < \varepsilon$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$. But then

$$\Bigl| |\varphi(\mathbf{x})| - |\varphi(\mathbf{p})| \Bigr| \leq |\varphi(\mathbf{x}) - \varphi(\mathbf{p})| < \varepsilon$$

for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$, and thus the function $|\varphi|$ is continuous, as required. $\blacksquare$

5.2. Continuous Functions and Open Sets

Let X and Y be subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$, and let $\varphi\colon X \to Y$ be a function from X to Y. We recall that the function φ is continuous at a point $\mathbf{p}$ of X if and only if, given any positive real number ε, there exists some positive real number δ such that $|\varphi(\mathbf{x}) - \varphi(\mathbf{p})| < \varepsilon$ for all points $\mathbf{x}$ of X satisfying $|\mathbf{x} - \mathbf{p}| < \delta$. Thus the function $\varphi\colon X \to Y$ is continuous at $\mathbf{p}$ if and only if, given any positive real number ε, there exists some positive real number δ such that the function φ maps the open ball $B_X(\mathbf{p}, \delta)$ in X of radius δ centred on the point $\mathbf{p}$ into the open ball $B_Y(\mathbf{q}, \varepsilon)$ in Y of radius ε centered on the point $\mathbf{q}$, where $\mathbf{q} = \varphi(\mathbf{p})$.

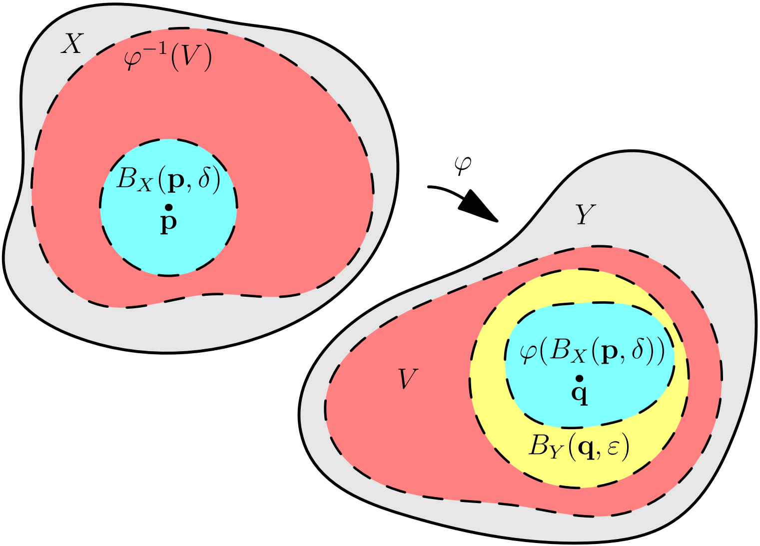
Given any function $\varphi \colon X \to Y$, we denote by $\varphi^{-1}(V)$ the *preimage* of a subset V of Y under the map φ, defined so that $\varphi^{-1}(V) = \{\mathbf{x} \in X : \varphi(\mathbf{x}) \in V\}$.

Proposition 5.7

Let X and Y be subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$, and let $\varphi\colon X \to Y$ be a function from X to Y. The function φ is continuous if and only if $\varphi^{-1}(V)$ is open in X for every open subset V of Y.

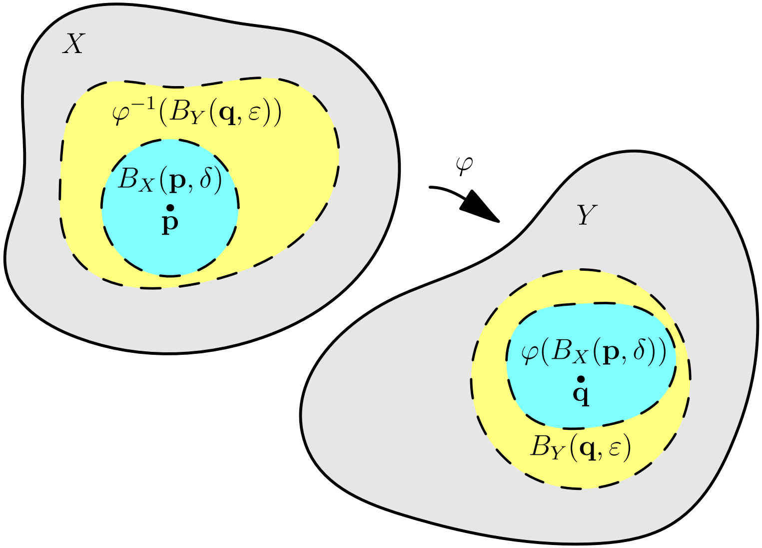
Proof

Suppose that $\varphi\colon X \to Y$ is continuous. Let V be an open set in Y. We must show that $\varphi^{-1}(V)$ is open in X. Let $\mathbf{p}$ be a point of $\varphi^{-1}(V)$, and let $\mathbf{q} = \varphi(\mathbf{p})$. Then $\mathbf{q} \in V$. But V is open, hence there exists some positive real number ε with the property that $B_Y(\mathbf{q}, \varepsilon) \subset V$. But φ is continuous at $\mathbf{p}$. Therefore there exists some positive real number δ such that φ maps $B_X(\mathbf{p}, \delta)$ into $B_Y(\mathbf{q}, \varepsilon)$. Thus $\varphi(\mathbf{x}) \in V$ for all $\mathbf{x} \in B_X(\mathbf{p}, \delta)$, showing that $B_X(\mathbf{p}, \delta) \subset \varphi^{-1}(V)$. This shows that $\varphi^{-1}(V)$ is open in X for every open set V in Y.



X
$\varphi^{-1}(V)$
$B_X(\mathbf{p}, \delta)$
$\mathbf{p}$
φ
Y
V
$\varphi(B_X(\mathbf{p}, \delta))$
$\mathbf{q}$
$B_Y(\mathbf{q}, \varepsilon)$

Conversely suppose that $\varphi\colon X \to Y$ is a function with the property that $\varphi^{-1}(V)$ is open in X for every open set V in Y. Let $\mathbf{p} \in X$, and let $\mathbf{q} = \varphi(\mathbf{p})$. We must show that φ is continuous at $\mathbf{p}$. Let some positive real number ε be given. Then $B_Y(\mathbf{q}, \varepsilon)$ is an open set in Y, by Lemma 4.1, hence $\varphi^{-1}(B_Y(\mathbf{q}, \varepsilon))$ is an open set in X which contains $\mathbf{p}$. It follows that there exists some positive real number δ such that $B_X(\mathbf{p}, \delta) \subset \varphi^{-1}(B_Y(\mathbf{q}, \varepsilon))$. Thus, given any positive real number ε, there exists some positive real number δ such that φ maps $B_X(\mathbf{p}, \delta)$ into $B_Y(\mathbf{q}, \varepsilon)$. We conclude that φ is continuous at the point $\mathbf{p}$, as required. $\blacksquare$


X
$\varphi^{-1}(B_Y(\mathbf{q}, \varepsilon))$
$B_X(\mathbf{p}, \delta)$
$\mathbf{p}$
φ
Y
$\varphi(B_X(\mathbf{p}, \delta))$
$\mathbf{q}$
$B_Y(\mathbf{q}, \varepsilon)$

Let X be a subset of $\mathbb{R}^n$, let $f: X \to \mathbb{R}$ be continuous, and let c be some real number. Then the sets

$$\{\mathbf{x} \in X : f(\mathbf{x}) > c\}$$

and

$$\{\mathbf{x} \in X : f(\mathbf{x}) < c\}$$

are open in X, and, given real numbers a and b satisfying $a < b$, the set

$$\{\mathbf{x} \in X : a < f(\mathbf{x}) < b\}$$

is open in X.

Again let X be a subset of $\mathbb{R}^n$, let $f\colon X \to \mathbb{R}$ be continuous, and let c be some real number. Now a subset of X is closed in X if and only if its complement is open in X. Consequently the sets

$$\{\mathbf{x} \in X : f(\mathbf{x}) \leq c\}$$

and

$$\{\mathbf{x} \in X : f(\mathbf{x}) \geq c\},$$

being the complements in X of sets that are open in X, must themselves be closed in X. It follows that that set

$$\{\mathbf{x} \in X : f(\mathbf{x}) = c\},$$

being the intersection of two subsets X that are closed in X, must itself be closed in X.

5.3. The Multidimensional Extreme Value Theorem

Lemma 5.8

Let X be a closed bounded set in $\mathbb{R}^m$, and let $f\colon X \to \mathbb{R}$ be a continuous real-valued function defined on X. Suppose that the set of values of the function f on X is bounded below. Then there exists a point $\mathbf{u}$ of X such that $f(\mathbf{u}) \leq f(\mathbf{x})$ for all $\mathbf{x} \in X$.

Proof

Let

$$m = \inf\{f(\mathbf{x}) : \mathbf{x} \in X\}.$$

Then there exists an infinite sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ in X such that

$$f(\mathbf{x}_j) < m + \frac{1}{j}$$

for all positive integers j. It follows from the multidimensional Bolzano-Weierstrass Theorem (Theorem 3.5) that this sequence has a subsequence $\mathbf{x}_{k_1}, \mathbf{x}_{k_2}, \mathbf{x}_{k_3}, \ldots$ which converges to some point $\mathbf{u}$ of $\mathbb{R}^m$.

Now the point $\mathbf{u}$ belongs to X because X is closed (see Lemma 4.7). Also

$$m \leq f(\mathbf{x}_{k_j}) < m + \frac{1}{k_j}$$

for all positive integers j. It follows that $\lim_{j \to +\infty} f(\mathbf{x}_{k_j}) = m$. Consequently

$$f(\mathbf{u}) = f\left(\lim_{j \to +\infty} \mathbf{x}_{k_j}\right) = \lim_{j \to +\infty} f(\mathbf{x}_{k_j}) = m$$

(see Proposition 5.2). It follows therefore that $f(\mathbf{x}) \geq f(\mathbf{u})$ for all $\mathbf{x} \in X$, Thus the function f attains a minimum value at the point $\mathbf{u}$ of X, which is what we were required to prove. ∎

Lemma 5.9

Let X be a closed bounded set in $\mathbb{R}^m$, and let $\varphi\colon X \to \mathbb{R}^n$ be a continuous function mapping X into $\mathbb{R}^n$. Then there exists a positive real number M with the property that $|\varphi(\mathbf{x})| \leq M$ for all $\mathbf{x} \in X$.

Proof

Let $g \colon X \to \mathbb{R}$ be defined such that

$$g(\mathbf{x}) = \frac{1}{1 + |\varphi(\mathbf{x})|}$$

for all $\mathbf{x} \in X$. Now the real-valued function mapping each $\mathbf{x} \in X$ to $|\varphi(\mathbf{x})|$ is continuous (see Lemma 5.6) and quotients of continuous real-valued functions are continuous where they are defined (see Lemma 5.5). It follows that the function $g \colon X \to \mathbb{R}$ is continuous. Moreover the values of this function are bounded below by zero. Consequently there exists some point $\mathbf{w}$ of X with the property that $g(\mathbf{x}) \geq g(\mathbf{w})$ for all $\mathbf{x} \in X$ (see Lemma 5.8). Let $M = |\varphi(\mathbf{w})|$. Then $|\varphi(\mathbf{x})| \leq M$ for all $\mathbf{x} \in X$. The result follows. ∎

Theorem 5.10 (The Multidimensional Extreme Value Theorem)

Let X be a closed bounded set in $\mathbb{R}^m$, and let $f: X \to \mathbb{R}$ be a continuous real-valued function defined on X. Then there exist points $\mathbf{u}$ and $\mathbf{v}$ of X such that $f(\mathbf{u}) \leq f(\mathbf{x}) \leq f(\mathbf{v})$ for all $\mathbf{x} \in X$.

Proof

It follows from Lemma 5.9 that there exists positive real number M with the property that $-M \leq f(\mathbf{x}) \leq M$ for all $\mathbf{x} \in X$. Thus the set of values of the function f is bounded above and below on X. Consequently there exist points $\mathbf{u}$ and $\mathbf{v}$ where the functions f and $-f$ respectively attain their minimum values on the set X (see Lemma 5.8). The result follows. ∎

5.4. Uniform Continuity for Functions of Several Real Variables

Definition

Let X be a subset of $\mathbb{R}^m$. A function $\varphi\colon X \to \mathbb{R}^n$ from X to $\mathbb{R}^n$ is said to be *uniformly continuous* if, given any positive real number ε, there exists some positive real number δ (whose value does not depend on either $\mathbf{y}$ or $\mathbf{z}$) such that $|\varphi(\mathbf{y}) - \varphi(\mathbf{z})| < \varepsilon$ for all points $\mathbf{y}$ and $\mathbf{z}$ of X satisfying $|\mathbf{y} - \mathbf{z}| < \delta$.

Theorem 5.11

Let X be a subset of $\mathbb{R}^m$ that is both closed and bounded. Then any continuous function $\varphi\colon X \to \mathbb{R}^n$ is uniformly continuous.

Proof

Let some positive real number ε be given. Suppose that there did not exist any positive real number δ small enough to ensure that $|\varphi(\mathbf{y}) - \varphi(\mathbf{z})| < \varepsilon$ for all points $\mathbf{y}$ and $\mathbf{z}$ of the set X satisfying $|\mathbf{y} - \mathbf{z}| < \delta$. Then, for each positive integer j, there would exist points $\mathbf{u}_j$ and $\mathbf{v}_j$ in X such that $|\mathbf{u}_j - \mathbf{v}_j| < 1/j$ and $|\varphi(\mathbf{u}_j) - \varphi(\mathbf{v}_j)| \geq \varepsilon$. But the sequence $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots$ would be bounded, since X is bounded, and thus would possess a subsequence $\mathbf{u}_{k_1}, \mathbf{u}_{k_2}, \mathbf{u}_{k_3}, \ldots$ converging to some point $\mathbf{p}$ (Theorem 3.5). Moreover $\mathbf{p} \in X$, because X is closed in $\mathbb{R}^n$. The sequence $\mathbf{v}_{k_1}, \mathbf{v}_{k_2}, \mathbf{v}_{k_3}, \ldots$ would also converge to $\mathbf{p}$, because

$$\lim_{j \to +\infty} |\mathbf{v}_{k_j} - \mathbf{u}_{k_j}| = 0.$$

But then the sequences

$$\varphi(\mathbf{u}_{k_1}), \varphi(\mathbf{u}_{k_2}), \varphi(\mathbf{u}_{k_3}), \ldots$$

and

$$\varphi(\mathbf{v}_{k_1}), \varphi(\mathbf{v}_{k_2}), \varphi(\mathbf{v}_{k_3}), \ldots$$

would both converge to $\varphi(\mathbf{p})$, because φ is continuous (see Proposition 5.2). Therefore

$$\lim_{j \to +\infty} \left| \varphi(\mathbf{u}_{k_j}) - \varphi(\mathbf{v}_{k_j}) \right| = 0.$$

But, assuming that no positive real number δ could be found satisfying the stated requirements, the points $\mathbf{u}_j$ and $\mathbf{v}_j$ had been chosen for all positive integers j so that $|\mathbf{u}_j - \mathbf{v}_j| < 1/j$ and $|\varphi(\mathbf{u}_j) - \varphi(\mathbf{v}_j)| \geq \varepsilon$. Consequently $\varphi(\mathbf{u}_{k_j})$ and $\varphi(\mathbf{v}_{k_j})$ could not both converge to $\varphi(\mathbf{p})$ as j increases to infinity. Thus the assumption that no positive real number δ would have the required property would lead to a contradiction. We conclude therefore that, in order to avoid arriving at this contradiction, there must exist some positive real number δ such that $|\varphi(\mathbf{y}) - \varphi(\mathbf{z})| < \varepsilon$ for all points $\mathbf{y}$ and $\mathbf{z}$ of the set X satisfying $|\mathbf{y} - \mathbf{z}| < \delta$, as required. $\blacksquare$