# MAU23203: Analysis in Several Real Variables Michaelmas Term 2021 Disquisition VIII: A Version of Taylor's Theorem with Integral Remainder

David R. Wilkins



We state and prove a version of Taylor's Theorem, applicable to functions that are $k$ times differentiable and whose derivatives of order up to and including $k$ are continuous functions, where the remainder term expressing the difference between the sum of the first $k$ terms of the Taylor expansion of the function and the function itself is expressed in the form of an integral.

**Theorem A (Taylor's Theorem with Integral Remainder)** *Let $s$ and $h$ be real numbers, and let $f$ be a function whose first $k$ derivatives are continuous on an open interval containing $s$ and $s+h$. Then*

$$f(s+h) = f(s) + \sum_{n=1}^{k-1} \frac{h^n}{n!} f^{(n)}(s) + \frac{h^k}{(k-1)!} \int_0^1 (1-t)^{k-1} f^{(k)}(s+th)\,dt.$$

**Proof** Let

$$r_m(s,h) = \frac{h^m}{(m-1)!} \int_0^1 (1-t)^{m-1} f^{(m)}(s+th)\,dt$$

for $m = 1, 2, \ldots, k-1$. Then

$$r_1(s,h) = h\int_0^1 f'(s+th)\,dt = \int_0^1 \frac{d}{dt} f(s+th)\,dt = f(s+h) - f(s).$$

Let $m$ be an integer between 1 and $k-2$. It follows from the rule for Integration by Parts (Corollary 7.24) that

$$r_{m+1}(s,h) = \frac{h^{m+1}}{m!} \int_0^1 (1-t)^m f^{(m+1)}(s+th)\,dt$$

$$
\begin{aligned}
&= \frac{h^m}{m!}\int_0^1 (1-t)^m \frac{d}{dt}\left(f^{(m)}(s+th)\right)\,dt \\
&= \frac{h^m}{m!}\left[(1-t)^m f^{(m)}(s+th)\right]_0^1 \\
&\qquad - \frac{h^m}{m!}\int_0^1 \frac{d}{dt}\left((1-t)^m\right) f^{(m)}(s+th)\,dt \\
&= -\frac{h^m}{m!} f^{(m)}(s) + \frac{h^m}{(m-1)!}\int_0^1 (1-t)^{m-1} f^{(m)}(s+th)\,dt \\
&= r_m(s,h) - \frac{h^m}{m!} f^{(m)}(s).
\end{aligned}
$$

Thus

$$
r_m(s,h) = \frac{h^m}{m!} f^{(m)}(s) + r_{m+1}(s,h)
$$

for $m = 1, 2, \ldots, k-1$. It follows by induction on $k$ that

$$
\begin{aligned}
f(s+h) &= f(s) + \sum_{n=1}^{k-1} \frac{h^n}{n!} f^{(n)}(s) + r_k(s,h) \\
&= f(s) + \sum_{n=1}^{k-1} \frac{h^n}{n!} f^{(n)}(s) + \frac{h^k}{(k-1)!}\int_0^1 (1-t)^{k-1} f^{(k)}(s+th)\,dt,
\end{aligned}
$$

as required. ∎