MAU23203: Analysis in Several Real Variables
Michaelmas Term 2021
Disquisition VI: Examples applying Taylor's Theorem

David R. Wilkins

We present below some examples of the applications of Taylor's Theorem (Theorem 7.14) in order to obtain the standard power series representations of the exponential, sine and cosine functions, proving the power series that are the Taylor expansions of these familiar functions converge to the functions in question.

Example The exponential function $\exp\colon \mathbb{R} \to \mathbb{R}$ satisfies

$$\exp(x) = \sum_{n=0}^{+\infty} \frac{x^n}{n!}$$

for all real numbers x.

The derivative of the exponential function is the exponential function itself. It follows from Taylor's Theorem that

$$\exp x = \sum_{n=0}^{m} \frac{x^n}{n!} + \frac{x^{m+1}}{(m+1)!} \exp(\theta x)$$

for some real number θ satisfying $0 < \theta < 1$. It follows that

$$\left| \exp x - \sum_{n=0}^{m} \frac{x^n}{n!} \right| \leq b_{m+1}(x) \exp(|x|),$$

where

$$b_n(x) = \frac{|x|^n}{n!}$$

for all real numbers x and non-negative integers n. Note that $b_n(x) \geq 0$ for all real numbers x and non-negative integers n.

Let N be some positive integer satisfying $N \geq 2|x|$. If n is a positive integer satisfying $n \geq N$ then $n + 1 > 2|x|$, and therefore

$$b_{n+1}(x) = \frac{|x|}{n+1} \times b_n(x) < \tfrac{1}{2} b_n(x).$$

It follows that $0 \leq b_n(x) < \dfrac{1}{2^{n-N}} b_N(x)$ whenever $n \geq N$, and therefore $\lim\limits_{n \to +\infty} b_n(x) = 0$. Thus

$$\left| \exp x - \sum_{n=0}^{m} \frac{x^n}{n!} \right| \to 0$$

as $m \to +\infty$, and thus

$$\exp x = \lim_{m \to +\infty} \sum_{n=0}^{m} \frac{x^n}{n!} = \sum_{n=0}^{+\infty} \frac{x^n}{n!},$$

as required.

Example We show that the sine function $\sin\colon \mathbb{R} \to \mathbb{R}$ satisfies

$$\sin x = \sum_{k=0}^{+\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} \quad \text{and} \quad \cos x = \sum_{k=0}^{+\infty} \frac{(-1)^k x^{2k}}{(2k)!}$$

for all real numbers x.

Indeed the derivatives of the sine function are given by

$$\sin^{(2k)}(x) = (-1)^k \sin(x) \quad \text{and} \quad \sin^{(2k+1)}(x) = (-1)^k \cos(x)$$

for all positive integers k. It follows from Taylor's Theorem that, given any real number x, and given any non-negative integer m, there exists some θ satisfying $0 < \theta < 1$ such that

$$\sin x = \sum_{k=0}^{m} \frac{(-1)^k x^{2k+1}}{(2k+1)!} + \frac{(-1)^{m+1} x^{2m+3}}{(2m+3)!} \cos(\theta x)$$

(The value of θ will depend on x and m.) It follows that

$$\left| \sin x - \sum_{k=0}^{m} \frac{(-1)^k x^{2k+1}}{(2k+1)!} \right| \leq b_{2m+3}(x),$$

for all non-negative integers m, where $b_n(x) = |x|^n/n!$ for all real numbers x and non-negative integers n. But it was shown in the proof of Corollary **??** that $\lim\limits_{n\to+\infty} b_n(x) = 0$ for all real numbers x. It follows that

$$\sin x = \lim_{m\to+\infty} \sum_{n=0}^{m} \frac{(-1)^k x^{2k+1}}{(2k+1)!} = \sum_{n=0}^{+\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}.$$

Example We show that the cosine function $\cos\colon \mathbb{R} \to \mathbb{R}$ satisfies

$$\cos x = \sum_{k=0}^{+\infty} \frac{(-1)^k x^{2k}}{(2k)!}$$

for all real numbers x.

Now the derivatives of the cosine function are given by

$$\cos^{(2k)}(x) = (-1)^k \cos(x) \quad \text{and} \quad \cos^{(2k-1)}(x) = (-1)^k \sin(x)$$

for all positive integers k. Therefore, given any real number x, and given any non-negative integer m, there exists some θ satisfying $0 < \theta < 1$ such that

$$\cos x = \sum_{k=0}^{m} \frac{(-1)^k x^{2k}}{(2k)!} + \frac{(-1)^{k+1} x^{2m+2}}{(2k+2)!} \cos(\theta x)$$

But then

$$\left| \cos x - \sum_{n=0}^{m} \frac{(-1)^n x^{2n}}{(2n)!} \right| \leq b_{2m+2}(x),$$

where, as before, $b_n(x) = |x|^n/n!$ for all real numbers x and non-negative integers n. But $\lim\limits_{n\to+\infty} b_n(x) = 0$ for all real numbers x. It follows that

$$\cos x = \lim_{m\to+\infty} \sum_{n=0}^{m} \frac{(-1)^n x^{2n}}{(2n)!} = \sum_{n=0}^{+\infty} \frac{(-1)^n x^{2n}}{(2n)!},$$

as required.