MAU23203: Analysis in Several Real Variables
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Disquisition VII: Smooth Functions of a Single Real Variable

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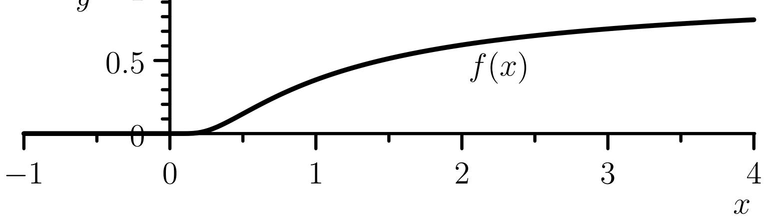
Definition A function of a single real variable is said to be *smooth* if it can be differentiated any number of times.

The class of smooth functions of a single real variable includes many familiar functions, such as polynomial functions, the exponential and logarithm functions, and trigonometrical functions such as the sine and cosines functions, and inverse trigonometrical functions.

Example Let $f\colon \mathbb{R} \to \mathbb{R}$ be the function mapping the set $\mathbb{R}$ of real numbers to itself defined such that

$$f(x) = \begin{cases} \exp\left(-\dfrac{1}{x}\right) & \text{if } x > 0; \\ 0 & \text{if } x \leq 0. \end{cases}$$

We show below that the function $f\colon \mathbb{R} \to \mathbb{R}$ is smooth on $\mathbb{R}$. In particular $f^{(k)}(0) = 0$ for all positive integers k.



First we show by induction on k that the function f is k times differentiable on $\mathbb{R}$ and $f^{(k)}(0) = 0$ for all positive integers k. Now it follows from standard rules for differentiating functions that

$$f^{(k)}(x) = \frac{p_k(x)}{x^{2k}} \exp\left(-\frac{1}{x}\right)$$

for all strictly positive real numbers x, where $p_1(x) = 1$ and

$$p_{k+1}(x) = x^2 p_k'(x) + (1 - 2kx)p_k(x)$$

for all k. A straightforward proof by induction shows that $p_k(x)$ is a polynomial in x of degree $k - 1$ for all positive integers k with leading term $(-1)^{k-1}k!x^{k-1}$.

Now

$$\frac{d}{dt}\left(t^n e^{-t}\right) = t^{n-1}(n - t)e^{-t}$$

for all positive real numbers t. It follows that function sending each positive real number t to $t^n e^{-t}$ is increasing when $0 \leq t < n$ and decreasing when $t > n$, and therefore $t^n e^{-t} \leq M_n$ for all positive real numbers t, where $M_n = n^n e^{-n}$. It follows that

$$0 \leq \frac{1}{x^{2k+1}} \exp\left(-\frac{1}{x}\right) \leq M_{2k+2}x$$

for all positive real numbers x, and therefore

$$\lim_{h \to 0^+} \frac{1}{h^{2k+1}} \exp\left(-\frac{1}{h}\right) = 0.$$

It then follows that

$$\begin{aligned} \lim_{h \to 0^+} \frac{f^{(k)}(h)}{h} &= \lim_{h \to 0^+} \left(\frac{p_k(h)}{h^{2k+1}} \exp\left(-\frac{1}{h}\right)\right) \\ &= \lim_{h \to 0^+} p_k(h) \times \lim_{h \to 0^+} \left(\frac{1}{h^{2k+1}} \exp\left(-\frac{1}{h}\right)\right) \\ &= p_k(0) \times 0 = 0 \end{aligned}$$

for all positive integers k. Now

$$\lim_{h \to 0^+} \frac{f(h) - f(0)}{h} = \lim_{h \to 0^+} \frac{f(h)}{h} = 0 = \lim_{h \to 0^-} \frac{f(h) - f(0)}{h}.$$

It follows that the function f is differentiable at zero, and $f'(0) = 0$.

Suppose that the function $f(x)$ is k-times differentiable at zero for some positive integer k, and that $f^{(k)}(0) = 0$. Then

$$\lim_{h \to 0^+} \frac{f^{(k)}(h) - f^{(k)}(0)}{h} = \lim_{h \to 0^+} \frac{f^{(k)}(h)}{h} = 0 = \lim_{h \to 0^-} \frac{f^{(k)}(h) - f^{(k)}(0)}{h}.$$

It then follows that the function $f^{(k)}$ is differentiable at zero, and moreover the derivative $f^{(k+1)}(0)$ of this function at zero is equal to zero. The function f is thus $(k+1)$-times differentiable at zero.

It now follows by induction on k that $f^{(k)}(x)$ exists for all positive integers k and real numbers x, and moreover

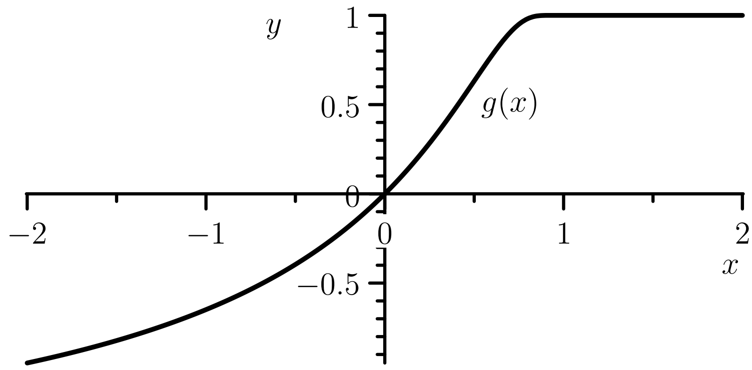
$$f^{(k)}(x) = \begin{cases} \dfrac{p_k(x)}{x^{2k}} \exp\left(-\dfrac{1}{x}\right) & \text{if } x > 0; \\ 0 & \text{if } x \leq 0. \end{cases}$$

The function $f\colon \mathbb{R} \to \mathbb{R}$ is thus a smooth function. Note however that the Taylor expansion of this function f about zero is the zero function. Consequently the function f discussed in this example provides an example of a smooth function that is not the sum of the Taylor series for that function about some particular point of its domain.

Example Let $g\colon \mathbb{R} \to \mathbb{R}$ be the function mapping the set $\mathbb{R}$ of real numbers to itself defined such that

$$g(x) = \begin{cases} 1 - \exp\left(-\dfrac{x}{1-x}\right) & \text{if } x < 1; \\ 1 & \text{if } x \geq 1. \end{cases}$$

We claim that the function $g\colon \mathbb{R} \to \mathbb{R}$ is smooth on $\mathbb{R}$. Moreover the function g is a strictly increasing function on $\{x \in \mathbb{R} : x < 1\}$, and $g(0) = 0$.



Let $f\colon \mathbb{R} \to \mathbb{R}$ be the real-valued function defined on the set $\mathbb{R}$ of real numbers so that

$$f(x) = \begin{cases} \exp\left(-\dfrac{1}{x}\right) & \text{if } x > 0; \\ 0 & \text{if } x \leq 0. \end{cases}$$

Now

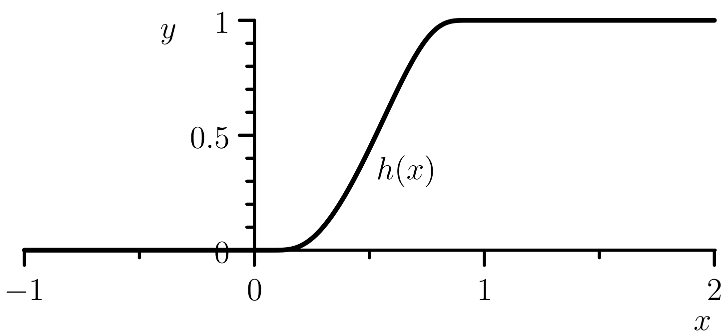
$$-\frac{x}{1-x} = 1 - \frac{1}{1-x}$$

for all real numbers x. It follows from the definition of the functions f and g that $g(x) = 1 - ef(1-x)$ for all real numbers x, where $e = \exp(1)$. Now we have already shown that the function f is smooth on the real line $\mathbb{R}$. It follows that the function g is also smooth on $\mathbb{R}$. Also $g(0) = 0$. Now $f(1-x)$ is a strictly decreasing function of x on $\{x \in \mathbb{R} : x < 1\}$. It follows that the function g is strictly increasing on that set, and thus has all the stated properties.

Example Let $h\colon \mathbb{R} \to \mathbb{R}$ be defined such that $h(x) = g(f(x)/f(1))$ for all real numbers x, where

$$\begin{aligned} f(x) &= \begin{cases} \exp\left(-\dfrac{1}{x}\right) & \text{if } x > 0, \\ 0 & \text{if } x \leq 0, \end{cases} \\ g(x) &= \begin{cases} 1 - \exp\left(-\dfrac{x}{1-x}\right) & \text{if } x < 0; \\ 1 & \text{if } x \geq 1. \end{cases} \end{aligned}$$

We claim that the function $h\colon \mathbb{R} \to \mathbb{R}$ is smooth, $h(x) = 0$ whenever $x \leq 0$, $h(1) = 1$ whenever $x \geq 1$, and $h(x)$ is a strictly increasing function of x when restricted to the interval $\{x \in \mathbb{R} : 0 < x < 1\}$.



Now the function h is a composition of smooth functions. Consequently applications of the Chain and Product Rules enable one to differentiate the function any number of times. The function h is therefore smooth. If $x \leq 0$ then $h(x) = g(f(0)) = g(0) = 0$. If $x \geq 1$ then $f(x)/f(1) \geq 1$ and therefore $h(x) = 1$. The function sending a real number x satisfying $0 < x < 1$ to $f(x)/f(1)$ is strictly increasing on the interval $(0, 1)$ and maps that interval into itself. Also the function g is strictly increasing on the interval $(0, 1)$. Thus the function h restricted to the interval $(0, 1)$ is a composition of two strictly increasing functions, and is thus itself strictly increasing.