MAU23203: Analysis in Several Real Variables
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Disquisition II: On Open Balls and Open Sets

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Open Balls and Sets in Euclidean Spaces

Given any point $\mathbf{p}$ of a Euclidean space $\mathbb{R}^n$, and given any positive real number η, the open ball of radius η centred on the point $\mathbf{p}$ consists of all points $\mathbf{x}$ of the Euclidean space $\mathbb{R}^n$ that lie within a distance η of the given point $\mathbf{p}$. Thus if $\mathbf{p} = (p_1, p_2, \ldots, p_n)$ then the *open ball* of radius η in $\mathbb{R}^n$ centred on the point $\mathbf{p}$ is the set

$$\{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : (x_1 - p_1)^2 + (x_2 - p_2)^2 + \cdots + (x_n - p_n)^2 < \eta^2\}.$$

A subset V of $\mathbb{R}^n$ is then said to be *open* in $\mathbb{R}^n$ if, given any point $\mathbf{p}$ of the set V, there exists some positive real number η for which the open ball of radius η centred on the point $\mathbf{p}$ is contained in the set V. A subset F of $\mathbb{R}^n$ of $\mathbb{R}^n$ is said to be *closed* in $\mathbb{R}^n$ if and only if its complement $\mathbb{R}^n \setminus F$ is open in $\mathbb{R}^n$. Here

$$\mathbb{R}^n \setminus F = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} \notin F\}.$$

Open balls in Euclidean spaces are open sets. This follows as an almost immediate consequence of the Triangle Inequality.

Open Disks and Open Sets in the Euclidean Plane

When working with subsets of the Euclidean plane $\mathbb{R}^2$, the terms *open ball* and *open disk* should be regarded as being synonymous: the term *open ball* applies to appropriate subsets of a Euclidean space of any dimension; but if the dimension of the space is equal to two, then the term *open disk* accords more closely with everyday usage outside mathematics.

Accordingly, the *open disk* in $\mathbb{R}^2$ of radius η centred on a point (a, b) of $\mathbb{R}^2$ is the set

$$\{(x, y) \in \mathbb{R}^2 : (x - a)^2 + (y - b)^2 = \eta\}.$$

Of course the equation right-hand side here can be manipulated in various ways. For example, the open disk of radius η in $\mathbb{R}^2$ centred on the point (a, b) may be represented as the set

$$\{(x, y) \in \mathbb{R}^2 : x^2 - 2ax + y^2 - 2by = \eta^2 - a^2 - b^2\}.$$

In particular, the open disks in $\mathbb{R}^2$ of radius 2 centred on the points $(1, 0)$ and $(-1, 0)$ are the sets

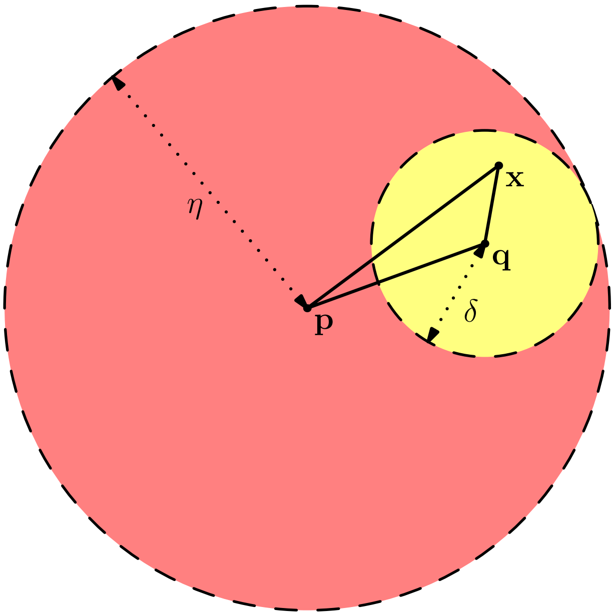
$$\{(x, y) \in \mathbb{R}^2 : x^2 - 2x + y^2 = 3\} \text{ and } \{(x, y) \in \mathbb{R}^2 : x^2 + 2x + y^2 = 3\}$$

respectively.

A subset of the plane $\mathbb{R}^2$ is open in $\mathbb{R}^2$ if and in only if, given any point of that subset, there exists some open disk centred on that point which is contained within the subset.

We now consider the basic result that open disks in $\mathbb{R}^2$ are open subsets of $\mathbb{R}^2$.

We consider an open disk of radius η centred on some given point $\mathbf{p}$, and take an arbitrary point $\mathbf{q}$ of this disk. Then $|\mathbf{q} - \mathbf{p}| < \eta$, where $|\mathbf{q} - \mathbf{p}|$ denotes the standard Euclidean distance between the points $\mathbf{p}$ and $\mathbf{q}$, this distance being the length, or norm, of the displacement vector $\mathbf{q} - \mathbf{p}$ between the points $\mathbf{p}$ and $\mathbf{q}$. We claim that if $\delta = \eta - |\mathbf{q} - \mathbf{p}|$ then the open disk of radius δ centred on the point $\mathbf{q}$ is contained in the open disk of radius η centred on the point $\mathbf{p}$. Accordingly we take an arbitrary point $\mathbf{x}$ of the open disk of radius δ centred on the point $\mathbf{q}$.



Now, in the triangle with vertices at the points $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{x}$, the length of the side with endpoints $\mathbf{p}$ and $\mathbf{x}$ is less than or equal to the sum of the lengths of the other two sides. In vector notation this basic inequality may be written as follows:

$$|\mathbf{x} - \mathbf{p}| \leq |\mathbf{x} - \mathbf{q}| + |\mathbf{q} - \mathbf{p}|.$$

Now, the point $\mathbf{x}$ is an arbitrary element of the open disk of radius δ centred on the point $\mathbf{q}$. This point therefore lies within a distance δ of the point $\mathbf{q}$. The Triangle Inequality stated above therefore ensures that

$$|\mathbf{x} - \mathbf{p}| < \delta + |\mathbf{q} - \mathbf{p}|.$$

The manner in which the positive real number δ has been chosen then ensures that $|\mathbf{x} - \mathbf{p}| < \eta$, thereby ensuring that the point $\mathbf{x}$ belongs to the open disk of radius η centred on the point $\mathbf{p}$.

It should be clear from the argument just presented that the basic result that open disks in the plane are open sets does not require the use of vector notation to establish its validity. The validity of this result follows from the fact that there is a well-defined notion of distance, determining in numerical terms the separation between any two points of the plane, where the distance function measuring the separation between points of the plane satisfies the Triangle Equality.

A proof of the result that the length of any side of a triangle is less than or equal to the sum of the lengths of the other two sides was given by Euclid in the 20th Proposition of the First Book of his *Elements of Geometry.* The proof given by Euclid is, in essentials, the same as that of the course in geometry included in the Junior and Leaving Certificate curricula in Ireland.

By far the most important commentary on the First Book of Euclid's *Elements of Geometry* surviving from ancient times is that of Proclus (410–485 C.E.), a Neo-Platonist philosopher and teacher of mathematics, who for many years was the director of the Academy in Athens. Proclus's commentary on the 20th Proposition of the First Book of the *Elements* begins as follows (in the translation published by Thomas Taylor in 1792):

> The Epicureans oppose the present theorem, asserting that it is manifest even to an ass; and that it requires no demonstration: and besides this, that it is alike the employment of the ignorant, to consider things manifest as worthy of proof, and to assent to such as are of themselves immanifest and unknown; for he who confounds these, seems to be ignorant of the difference between demonstrable and indemonstrable. But that the present theorem is known even to an ass, they evince from hence, that grass being placed in one extremity of the sides, the ass seeking his food, wanders over one side, and not over two. Against these we reply, that the present theorem is indeed manifest to sense, but not to reason producing science: for this is the case in a variety of concerns. Thus for example, we are indubitably certain from sense, that fire warms, but it is the business of science to convince us how it warms; whether by an incorporeal power, or by corporeal sections; whether by spherical, or pyramidal particles. Again, that we are moved is evident to sense, but it is difficult to assign a rational cause how we are moved; whether over an impartible, or over an interval: but how can we run through infinite, since every magnitude is divisible in infinitum? Let, therefore, the present theorem, that the two sides of a triangle are greater than the remainder, be manifest to sense, yet it belongs to science to inform

> us how this is effected. And thus much may suffice against the Epicureans.

(Regarding the suggestion that fire might warm by “pyramidal particles”, Proclus was here clearly making reference to the account of the nature of the earth and the cosmos, presented by Plato in his *Timæus* dialogue, that made the claim that the four elements Fire, Air, Water and Earth recognized by ancient Greek philosophers were constituted of particles that were respectively tetrahedral, octahedral, icosahedral and cubic in form.)

Open Disks in Euclidean Spaces and Metric Spaces

Resuming the discussion of open sets, it should be noted that the principle that open disks are open sets in the plane generalizes to higher dimensions. Imagine the surface of the Earth (or some idealized Earth) as being a perfect sphere. This sphere then bounds a ball. If some point is located 100 kilometres below the surface of the (perfectly spherical) Earth, then the open ball of radius 100 kilometres centered on that point in the interior of the Earth is wholly contained in the interior of the Earth.

Ultimately the proposition generalizes so as to apply to open balls and open sets in metric spaces. A *metric space* consists of a set X, together with a distance function d determining in numerical terms the separation or distance $d(x, y)$ between any two points x and y of the set X of the set, where the distance function is required to satisfy the following four axioms:—

(i) $d(x, y) \geq 0$ for all points x and y of X;

(ii) $d(x, y) = d(y, x)$ for all points x and y of X;

(iii) $d(x, z) \leq d(x, y) + d(y, z)$ for all points x, y and z of X;

(iv) points x and y of X satisfy $d(x, y) = 0$ if and only if $x = y$.

The definitions of *open ball* and *open set*, applicable to subsets of any metric space, generalize in the most direct and obvious fashion the standard definitions of open ball and open set applicable to subsets of Euclidean spaces. The validity of the Triangle Inequality is built into the very axioms which the distance function on a metric space must satisfy. Consequently the proof of the result that open balls are open sets also generalizes in the most direct and obvious fashion so as to apply to open balls and open sets in any metric space.

Indeed let X be a metric space with distance function d, let p be a point of X and let η be a positive real number. The open ball of radius η centred on the point p by definition consists of those points q of the metric space for which $d(p,q) < \eta$. Accordingly let q be an arbitrary point of the open ball in X centred on the point p, and let $\delta = \eta - d(p,q)$. Then $\delta > 0$. Moreover if $x \in X$ satisfies $d(x,q) < \delta$ then, by the version of the Triangle Inequality included in the metric space axioms (see (iii) above),

$$d(p,x) \leq d(p,q) + d(q,x) < d(p,q) + \delta = \eta,$$

and therefore the arbitrarily chosen point x of open ball of radius δ about the point q belongs also to the open ball of radius η about the point p. Thus the open ball of radius δ centred on the point q is contained in the open ball of radius η centred on the point p.

Now a subset V of the metric space X is said to be *open* in X if, given any point of V, there exists some open ball in X centred on that point which is wholly contained within the set V. It follows from the argument just presented that any open ball in a metric space is an open set in that space.

Regions Exterior to Spheres and Circles

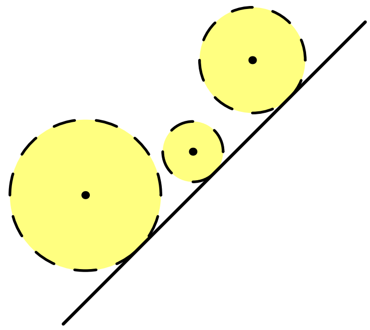
Let us now discuss the regions of Euclidean space that are exterior to spheres. Such regions are also open sets. Let us consider the three-dimensional case. Again consider the surface of the Earth to be a perfect sphere. Let an aeroplane happen to be one kilometre above the surface of the Earth. Then any bird located within a distance of one kilometre of the aeroplane must be airborne. Indeed suppose it were the case that the bird were located in the interior of the Earth, or on the surface of the Earth. (For example, maybe the bird in question was a deceased parrot that got chucked down a well and subsequently buried under a pile of rubble.) Consider the triangle with vertices at the aeroplane, the bird and the centre of the Earth. The distance from the aeroplane to the centre of the Earth would exceed the radius of the Earth by one kilometre. Now if the bird were both located in the interior of the Earth, or on the surface of the Earth, and were also within a distance of one kilometre from the plane, then the sum of the distances from the centre of the Earth to the bird, and from the bird to the aeroplane would be strictly less than the direct distance from the centre of the Earth to the aeroplane, contradicting the Triangle inequality. It follows, as claimed, that if a bird is located within a distance of one kilometre from the aeroplane then that bird must be airborne. Or, in other words, the open ball centred on the aeroplane is contained wholly within the atmosphere, provided that the radius of the ball does not exceed one kilometre. And similarly if the aeroplane were cruising ten kilometres above the surface of the Earth then a ball centred on the aeroplane is contained wholly within the atmosphere, provided that the radius of the ball does not exceed ten kilometres. Thus, if the areoplane were taking off or landing, then, at any time whilst the aeroplane is airborne, an open ball whose radius less than or equal to the height of the aeroplane at that time would be wholly contained within the atmosphere.

The region outside a sphere in three-dimensional space is accordingly an open set in three-dimensional space. Similarly the region outside a circle in the plane is an open set in that plane. Given a point lying outside the circle, one can consider the open disk centred on the given point whose radius is determined by subtracting the radius of the circle from the distance of the given point from the centre of the circle. Such an open disk lies wholly outside the circle. And the formal proof of this result utilizes the standard Triangle Inequality.

Open Half-Spaces and Half-Planes

Returning to the aeroplane metaphor, now suppose that some portion of the Earth's surface is a perfectly flat featureless prairie. If an aeroplane is located one kilometre above that prairie then any bird located within a distance of one kilometre from the aeroplane is airborne. And similarly, whenever the aeroplane is itself airborne, any bird whose distance from the aeroplane were less than the height of the aeroplane above the surface of the prairie would be airborne. Moreover the open ball centred on the areoplane is wholly contained within the atmosphere provided that the radius of the ball is less than or equal to the height of the aeroplane above the prairie.

These considerations illustrate the mathematical result that a complement of a plane in three-dimensional space consists of two half-spaces, each of which is an open subset of three-dimensional space. Similarly the complement of any unbounded straight line in the plane consists of two open half-planes, each of which is an open subset of the plane. Indeed if we take a point of the plane that does not lie on the line then the open disk whose radius is the perpendicular distance from the point to the line will be wholly contained within that side of the line to which the point belongs.

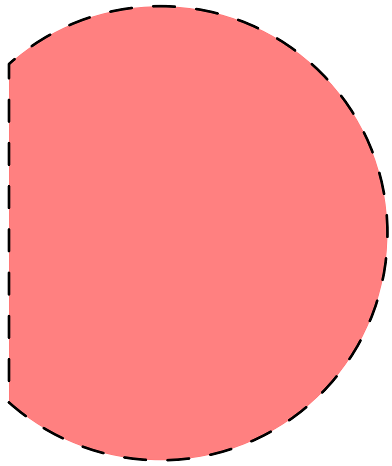


Intersecting Disks and Half-Planes

Let V be the subset of the plane defined so that

$$V = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 9 \text{ and } x + 2 > 0\}.$$

This set is pictured below.



This set V is an open set in the plane $\mathbb{R}^2$. In fact, given an arbitrary point (a, b) of the set, the open disk of radius δ centred on the point (a, b) is contained within the set, provided that

$$0 < \delta \leq \min(3 - \sqrt{a^2 + b^2},\, a + 2).$$

In geometric terms, the maximum value of δ associated with a point (a, b) of the set V is the distance from that point (a, b) to the boundary of the set V. Some points of the set V are closer to the straight part of the boundary; other points the set V are closer to the circular arc that is the portion of the circle $x^2 + y^2 = 9$ included in the boundary of the set V. However, if one were asked to justify the assertion that the set V is open in the plane, a lengthy explanation to establish an explicit range for permissible values of the radius of open disks fitting within the set would provide unnecessary detail. A more succinct method of justifying the assertion would note first that the set V is the intersection of two open sets, the first of those open sets being the open disk of radius 3 centred on the origin $(0, 0)$ and the second being the open-half space consisting of all all points (x, y) of the plane for which $x > -2$. Now any finite intersection of open sets in the plane must itself be an open set. Therefore, in particular, the set V, being the intersection of two open sets, must itself be an open set.

One can also consider whether or not the set V is a closed set. It is not. To justify the assertion that the set V is not closed, it suffices to show that the complement of the set is not open. And to show that the complement of the set V is open, one should exhibit a point of the complement of the set V that is not the centre of any open ball of positive radius contained in the complement of the set V. In other words, one should exhibit a point that belongs to the complement of the set V but also has the property that every open ball of positive radius centred on the exhibited point intersects the set V. Examples of points of the complement of the set V with this property are the following: the point $(-2, 0)$, the point $(3, 0)$, the point $(0, 3)$. Note that the set V has an obvious boundary, and the boundary of the set V is where one would seek points that do not belong to the set V but are limit points of the set V.