MAU23203: Analysis in Several Real Variables
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Disquisition III: Open and Closed Set Examples

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Examples involving Intersections of Disks with Half-Planes

We have already considered the nature of the subset V of the plane defined so that

$$V = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 9 \text{ and } x + 2 > 0\}.$$

This set is an open set in the plane. It is not a closed set.

We consider some related examples.

Example Let M be the subset of the plane $\mathbb{R}^2$ defined so that

$$M = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 9 \text{ and } x + 2 \geq 0\}.$$

We consider whether or not this set is open in the plane, and whether or not it is closed in the plane.

Now the set M is the intersection of an open set and a closed set. Specifically it is the intersection of the open ball

$$\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 9\}$$

of radius 3 centred on the origin and the closed half-space

$$\{(x, y) \in \mathbb{R}^2 : x \geq -2\}.$$

This observation allows for the possibility that the set M may be neither open nor closed. However it does not resolve the question as to whether or not the set is open, or that as to whether or not the set is closed. One can construct examples each arising as an intersection of an open set with a closed set where that intersection is an open set, and one can construct other examples where the intersection of an open set and a closed set is a closed set.

As it happens, the set M specified about is neither open nor closed in $\mathbb{R}^2$. For example the point $(-2, 0)$ belongs to the set M, but any open ball of positive radius about the point $(-2, 0)$ contains points (x, y) for which $x < -2$, and consequently no open ball of positive radius can be found centred on the point $(-2, 0)$ that is wholly contained within the set M. Thus the set M is not an open set.

Also consider the point $(3, 0)$. This point does not belong to the set M. But an open ball of radius δ centred on the point $(3, 0)$ contains all points $(x, 0)$ for which $3-\delta < x < 3$ and $x > 0$, and such points belong to the set M. Therefore the complement of the set M is not an open set, and therefore the set M itself is not a closed set.

Now, in order to demonstrate that the set M is not closed, we chose to consider open balls centered on the point $(3, 0)$. Alternatively we could have considered open balls centred on the point $(0, 3)$, or centred on the point $(0, -3)$. Indeed we could have considered open balls centred on any point of the circular arc that consists of those points lying along the boundary of the set M that do not belong to M itself. Similarly to show that the set M is not open we could have considered open balls centred at any point of the form $(-2, b)$ for which $|b| < \sqrt{5}$. Those points are the points lying on the boundary of the set M that belong to the set M itself.

Example Let W be the subset of the plane $\mathbb{R}^2$ defined so that

$$W = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 9 \text{ or } x + 2 > 0\}.$$

The set W is an open set in the plane $\mathbb{R}^2$ because it is the union of the open ball of radius 3 centred on the origin $(0, 0)$ and the open half-space consisting of those points (x, y) of the plane for which $x > -2$.

In order to show that the set W is not closed, we can consider open balls about the point $(-3, 0)$. The point $(-3, 0)$ does not belong to the set W, but any open ball of radius δ centred on the point $(-3, 0)$ contains some points $(x, 0)$ of the plane $\mathbb{R}^2$ for which $-3 < x < -3 + \delta$ and $x < 0$, and such points $(x, 0)$ belong to the set W. Therefore the complement of the set W is not an open set in $\mathbb{R}^2$, and therefore the set W itself is not a closed set in $\mathbb{R}^2$.

In order to show that the set W is not closed, we can consider open balls about any point (a, b) for which $a^2 + b^2 = 9$ and $a \leq -2$. We can also consider open balls about any point $(-2, b)$ for which $|b| \geq \sqrt{5}$. These are the points that lie along the obvious boundary of the set W: none of those points belongs to the set W itself.

Examples involving Intersections of Two Disks

Example Let Y be the subset of the plane $\mathbb{R}^2$ defined so that

$$Y = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 8 - 2x \text{ and } x^2 + y^2 < 8 + 2x\}.$$

Now $x^2 + y^2 < 8 - 2x$ if and only if $(x + 1)^2 + y^2 < 9$, and $x^2 + y^2 < 8 + 2x$ if and only if $(x - 1)^2 + y^2 < 9$. It follows that the set Y is the intersection of open balls of radius 3 centred on the points $(-1, 0)$ and $(1, 0)$. Consequently the set Y is an open set.

The set Y is not a closed set. Indeed the point $(2, 0)$ belongs to the complement of the set, but every open ball of positive radius centred on this point intersects the set.

Examples involving Intersections of Two Disks

Example Let N be the subset of the plane $\mathbb{R}^2$ defined so that

$$N = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 8 - 2x \text{ and } x^2 + y^2 < 8 + 2x\}.$$

Now $x^2 + y^2 \leq 8 - 2x$ if and only if $(x + 1)^2 + y^2 \leq 9$, and $x^2 + y^2 < 8 + 2x$ if and only if $(x - 1)^2 + y^2 < 9$. It follows that the set Y is the intersection of the closed ball of radius 3 centred on the point $(-1, 0)$ and the open ball of radius 3 centred on the point $(1, 0)$.

The set N is not an open set. Indeed the point $(2, 0)$ belongs to the set, but no open ball of positive radius centered on that point is contained wholly within the set.

The set N is not a closed set. Indeed the point $(-2, 0)$ belongs to the complement of the set, but every open ball of positive radius centred on this point intersects the set. Therefore the complement of the set N is not open, and therefore the set N itself is not closed.

Examples involving Disks and Annuli

Example Let

$$A = \{(x, y) \in \mathbb{R}^2 : 1 < x^2 + y^2 < 9\} \quad \text{and} \quad B = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 4\}.$$

The set A itself is an open set. Indeed this set is the intersection of the open ball of radius 3 about the origin with the open subset of the plane consisting of those points of the plane that lie outside the circle of radius 1 centred on the origin. The set A is thus the intersection of two open sets, and is thus itself an open set.

The set B is the closed ball of radius 2 centred on the origin. It is therefore a closed set in $\mathbb{R}^2$.

The union of the sets A and B is the open ball of radius 3 centred on the origin. Thus the union of the sets A and B is an open ball. We have thus exhibited an example where the union of an open set and a closed set is an open set. Moreover neither set is contained within the other.

Example In the case of the sets

$$\{(x, y) \in \mathbb{R}^2 : 1 \leq x^2 + y^2 \leq 9\} \quad \text{and} \quad \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 4\},$$

the first of the two sets is closed, the second is open and the union of the two sets is closed.

Example In the case of the sets

$$\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1 \text{ or } x^2 + y^2 > 9\} \quad \text{and} \quad \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \geq 4\},$$

the first of the two sets is open, the second is closed and the intersection of the two sets is closed.

Example In the case of the sets

$$\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1 \text{ or } x^2 + y^2 \geq 9\} \quad \text{and} \quad \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 > 4\},$$

the first of the two sets is closed, the second is open and the intersection of the two sets is open.