MAU23203: Analysis in Several Real Variables
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Disquisition IX: Multiple Integrals

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We consider integrals of continuous real-valued functions of several real variables over regions that are products of closed bounded intervals. Any subset of n-dimensional Euclidean space $\mathbb{R}^n$ that is a product of closed bounded intervals is a closed bounded set in $\mathbb{R}^n$. It follows from the Extreme Value Theorem (Theorem 5.10) that any continuous real-valued function on a product of closed bounded intervals is necessarily bounded on that product of intervals. It is also uniformly continuous on that product of intervals (see Theorem 5.11)

Proposition A *Let n be an integer greater than 1, let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be real numbers, where $a_i < b_i$ for $i = 1, 2, \ldots, n$, let $f\colon [a_1, b_1] \times \cdots \times [a_n, b_n] \to \mathbb{R}$ be a continuous real-valued function, and let*

$$g(x_1, x_2, \ldots, x_{n-1}) = \int_{a_n}^{b_n} f(x_1, x_2, \ldots, x_{n-1}, t)\, dt.$$

for all $(n-1)$-tuples $(x_1, x_2, \ldots, x_{n-1})$ of real numbers satisfying $a_i \leq x_i \leq b_i$ for $i = 1, 2, \ldots, n-1$. Then the function

$$g\colon [a_1, b_1] \times [a_2, b_2] \cdots \times [a_{n-1}, b_{n-1}] \to \mathbb{R}$$

is continuous.

Proof Let some positive real number ε be given, and let ε_0 be chosen so that $0 < (b_n - a_n)\varepsilon_0 < \varepsilon$. The function f is uniformly continuous on $[a_1, b_1] \times [a_2, b_2] \cdots \times [a_n, b_n]$ (see Theorem 5.11). Therefore there exists some positive real number δ such that

$$|f(x_1, x_2, \ldots, x_{n-1}, t) - f(u_1, u_2, \ldots, u_{n-1}, t)| < \varepsilon_0$$

for all real numbers $x_1, x_2, \ldots, x_{n-1}$, $u_1, u_2, \ldots, u_{n-1}$ and t satisfying $a_i \leq x_i \leq b_i$, $a_i \leq u_i < b_i$ and $|x_i - u_i| < \delta$ for $i = 1, 2, \ldots, n-1$ and $a_n \leq t \leq b_n$. Then

$$\begin{aligned}
&|g(x_1, x_2, \ldots, x_{n-1}) - g(u_1, u_2, \ldots, u_{n-1})| \\
&\qquad = \left| \int_{a_n}^{b_n} (f(x_1, x_2, \ldots, x_{n-1}, t) - f(u_1, u_2, \ldots, u_{n-1}, t))\, dt \right| \\
&\qquad \leq \int_{a_n}^{b_n} |f(x_1, x_2, \ldots, x_{n-1}, t) - f(u_1, u_2, \ldots, u_{n-1}, t)|\, dt \\
&\qquad \leq \varepsilon_0 (b_n - a_n) < \varepsilon
\end{aligned}$$

whenever $a_i \leq x_i \leq b_i$, $a_i \leq u_i < b_i$ and $|x_i - u_i| < \delta$ for $i = 1, 2, \ldots, n-1$. The result follows. ∎

Proposition A ensures that, given a continuous real-valued function $f\colon [a_1, b_1] \times \cdots \times [a_n, b_n] \to \mathbb{R}$, where $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ are real numbers and $a_i < b_i$ for $i = 1, 2, \ldots, n$, there is a well-defined multiple integral

$$\int_{x_n = a_n}^{b_n} \cdots \int_{x_2 = a_2}^{b_2} \int_{x_1 = a_1}^{b_1} f(x_1, x_2, \ldots, x_n)\, dx_1\, dx_2 \cdots dx_n,$$

in which, at each stage of evaluation, the integrand is a continuous function of its arguments. To evaluate this integral, one integrates first with respect to x_1, then with respect to x_2, and so on, finally integrating with respect to x_n.

In fact, if the function f is continuous, the order of evaluation of the integrals with respect to the individual variables does not affect the value of the multiple integral. We prove this first for continuous functions of two variables.

Theorem B *Let $f\colon [a_1, b_1] \times [a_2, b_2] \to \mathbb{R}$ be a continuous real-valued function on the closed rectangle $[a_1, b_1] \times [a_2, b_2]$. Then*

$$\int_{a_2}^{b_2} \left(\int_{a_1}^{b_1} f(x, y)\, dx \right) dy = \int_{a_1}^{b_1} \left(\int_{a_2}^{b_2} f(x, y)\, dy \right) dx.$$

Proof Let $f\colon [a_1, b_1] \times [a_2, b_2] \to \mathbb{R}$ is continuous, and is therefore uniformly continuous on $[a_1, b_1] \times [a_2, b_2]$ (see Theorem 5.11). Let some positive real number ε be given. It follows from the uniform continuity of the function f that there exists some positive real number δ with the property that

$$|f(x, y) - f(u, v)| < \varepsilon$$

for all $x, u \in [a_1, b_1]$ and $y, v \in [a_2, b_2]$ satisfying $|x - u| < \delta$ and $|y - v| < \delta$. Let P be a partition of $[a_1, b_1]$, and let Q be a partition of $[a_2, b_2]$, where

$$P = \{u_0, u_1, \ldots, u_p\}, \quad Q = \{v_0, v_1, \ldots, v_q\},$$

$$a_1 = u_0 < u_1 < \cdots < u_p = b_1, \quad a_2 = v_0 < v_1 < \cdots < v_q = b_2,$$

$u_j - u_{j-1} < \delta$ for $j = 1, 2, \ldots, p$ and $v_k - v_{k-1} < \delta$ for $k = 1, 2, \ldots, q$. Then

$$|f(x, y) - f(u_j, v_k)| < \varepsilon$$

whenever $u_{j-1} \leq x \leq u_j$ for some integer j between 1 and p and $v_{k-1} \leq y \leq v_k$ for some integer k between 1 and q.

Now

$$\int_{a_2}^{b_2} \left(\int_{a_1}^{b_1} f(x, y)\, dx \right) dy = \sum_{k=1}^{q} \sum_{j=1}^{p} \int_{v_{k-1}}^{v_k} \left(\int_{u_{j-1}}^{u_j} f(x, y)\, dx \right) dy.$$

Moreover

$$\int_{u_{j-1}}^{u_j} f(x, y)\, dx \leq \Big(f(u_j, v_k) + \varepsilon)(u_j - u_{j-1})$$

for all $y \in [v_{k-1}, v_k]$, and therefore

$$\int_{v_{k-1}}^{v_k} \left(\int_{u_{j-1}}^{u_j} f(x, y)\, dx \right) dy \leq \Big(f(u_j, v_k) + \varepsilon \Big)(v_k - v_{k-1})(u_j - u_{j-1})$$

for all integers j between 1 and p and integers k between 1 and q. It follows that

$$\begin{aligned} \int_{a_2}^{b_2} \left(\int_{a_1}^{b_1} f(x, y)\, dx \right) dy &\leq \sum_{k=1}^{q} \sum_{j=1}^{p} \Big(f(u_j, v_k) + \varepsilon \Big)(v_k - v_{k-1})(u_j - u_{j-1}) \\ &= S + \varepsilon(b_1 - a_1)(b_2 - a_2), \end{aligned}$$

where

$$S = \sum_{k=1}^{q} \sum_{j=1}^{p} f(u_j, v_k)(v_k - v_{k-1})(u_j - u_{j-1}).$$

Similarly

$$\begin{aligned} \int_{a_2}^{b_2} \left(\int_{a_1}^{b_1} f(x, y)\, dx \right) dy &\geq \sum_{k=1}^{q} \sum_{j=1}^{p} \Big(f(u_j, v_k) - \varepsilon \Big)(v_k - v_{k-1})(u_j - u_{j-1}) \\ &= S - \varepsilon(b_1 - a_1)(b_2 - a_2). \end{aligned}$$

Thus

$$\left|\int_{a_2}^{b_2}\left(\int_{a_1}^{b_1} f(x,y)\,dx\right)\,dy - S\right| \le \varepsilon(b_1-a_1)(b_2-a_2).$$

On interchanging the roles of the variables x and y, we conclude similarly that

$$\left|\int_{a_1}^{b_1}\left(\int_{a_2}^{b_2} f(x,y)\,dy\right)\,dx - S\right| \le \varepsilon(b_1-a_1)(b_2-a_2).$$

It follows that

$$\begin{aligned}\left|\int_{a_2}^{b_2}\left(\int_{a_1}^{b_1} f(x,y)\,dx\right)\,dy - \int_{a_1}^{b_1}\left(\int_{a_2}^{b_2} f(x,y)\,dy\right)\,dx\right|\\ \le\ 2\varepsilon(b_1-a_1)(b_2-a_2).\end{aligned}$$

Moreover the inequality just obtained must hold for every positive real number ε, no matter how small the value of ε. It follows that

$$\int_{a_2}^{b_2}\left(\int_{a_1}^{b_1} f(x,y)\,dx\right)\,dy = \int_{a_1}^{b_1}\left(\int_{a_2}^{b_2} f(x,y)\,dy\right)\,dx,$$

as required. ∎

Now let us consider a multiple integral involving a continuous function of three real variables. Let

$$f\colon [a_1,b_1]\times[a_2,b_2]\times[a_3,b_3]\to\mathbb{R}$$

be a continuous real-valued function, where a_1, a_2, a_3, b_1, b_2 and b_3 are real numbers satisfying $a_1 < b_1$, $a_2 < b_2$ and $a_3 < b_3$. It follows from Theorem B that

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2} f(x_1,x_2,x_3)\,dx_2\,dx_1 = \int_{a_2}^{b_2}\int_{a_1}^{b_1} f(x_1,x_2,x_3)\,dx_1\,dx_2$$

for all real numbers x_3 satisfying $a_3 < x_3 < b_3$. It follows that

$$\begin{aligned}&\int_{a_3}^{b_3}\int_{a_1}^{b_1}\int_{a_2}^{b_2} f(x_1,x_2,x_3)\,dx_2\,dx_1\,dx_3\\ &\qquad = \int_{a_3}^{b_3}\int_{a_2}^{b_2}\int_{a_1}^{b_1} f(x_1,x_2,x_3)\,dx_1\,dx_2\,dx_3.\end{aligned}$$

Also it follows from Proposition A that the function sending (x_2,x_3) to

$$\int_{a_1}^{b_1} f(x_1,x_2,x_3)\,dx_1$$

for all $(x_2, x_3) \in [a_2, b_2] \times [a_3, b_3]$ is a continuous function of (x_2, x_3). It then follows from Theorem B that

$$\begin{aligned} &\int_{a_2}^{b_2} \int_{a_3}^{b_3} \int_{a_1}^{b_1} f(x_1, x_2, x_3)\, dx_1\, dx_3\, dx_2 \\ &\qquad = \int_{a_3}^{b_3} \int_{a_2}^{b_2} \int_{a_1}^{b_1} f(x_1, x_2, x_3)\, dx_1\, dx_2\, dx_3. \end{aligned}$$

Repeated applications of these results establish that the value of the repeated integral with respect to the real variables x_1, x_2 and x_3 is independent of the order in which the successive integrations are performed.

Corresponding results hold for integration of continuous real-valued functions of four or more real variables. In general, if the integrand is a continuous real-valued function of n real variables, and if this function is integrated over a product of n closed bounded intervals, by repeated integration, then the value of the integral is independent of the order in which the integrals are performed.

0.1 A Counterexample involving an Unbounded Function

Example Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be defined such that

$$f(x, y) = \begin{cases} \dfrac{4xy(x^2 - y^2)}{(x^2 + y^2)^3} & \text{if } (x, y) \neq (0, 0); \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

Set $u = x^2 + y^2$. Then

$$f(x, y) = \frac{2x(2x^2 - u)}{u^3} \frac{\partial u}{\partial y},$$

and therefore, when $x \neq 0$,

$$\begin{aligned} \int_{y=0}^{1} f(x, y)\, dy &= \int_{u=x^2}^{x^2+1} \left(\frac{4x^3}{u^3} - \frac{2x}{u^2} \right) du \\ &= \left[-\frac{2x^3}{u^2} + \frac{2x}{u} \right]_{u=x^2}^{x^2+1} \\ &= -\frac{2x^3}{(x^2+1)^2} + \frac{2x}{x^2+1} \\ &= \frac{2x}{(x^2+1)^2} \end{aligned}$$

It follows that

$$\begin{aligned}\int_{x=0}^{1}\left(\int_{y=0}^{1} f(x,y)\,dy\right)\,dx &= \int_{x=0}^{1}\frac{2x}{(x^2+1)^2}\,dx\\ &= \left[-\frac{1}{x^2+1}\right]_0^1 = \frac{1}{2}.\end{aligned}$$

Now $f(y,x) = -f(x,y)$ for all x and y. Interchanging x and y in the above evaluation, we find that

$$\begin{aligned}\int_{y=0}^{1}\left(\int_{x=0}^{1} f(x,y)\,dx\right)\,dy &= \int_{x=0}^{1}\left(\int_{y=0}^{1} f(y,x)\,dy\right)\,dx\\ &= -\int_{x=0}^{1}\left(\int_{y=0}^{1} f(x,y)\,dy\right)\,dx\\ &= -\frac{1}{2}.\end{aligned}$$

Thus

$$\int_{x=0}^{1}\left(\int_{y=0}^{1} f(x,y)\,dy\right)\,dx \neq \int_{y=0}^{1}\left(\int_{x=0}^{1} f(x,y)\,dx\right)\,dy.$$

when

$$f(x,y) = \frac{4xy(x^2-y^2)}{(x^2+y^2)^3}$$

for all $(x,y) \in \mathbb{R}^2$ distinct from $(0,0)$. Note that, in this case $f(2t,t) \to +\infty$ as $t \to 0^+$, and $f(t,2t) \to -\infty$ as $t \to 0^-$. Thus the function f is not continuous at $(0,0)$ and does not remain bounded as $(x,y) \to (0,0)$.