MAU23203: Analysis in Several Real Variables
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Disquisition I: Convergent Subsequence Examples

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An Example concerning Convergent Subsequences of a Bounded Infinite Sequence

We recall that an infinite sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ of points in n-dimensional Euclidean space $\mathbb{R}^n$ is said to be *bounded* if there exists some constant K such that $|\mathbf{x}_j| \leq K$ for all j.

The multidimensional Bolzano-Weierstrass Theorem asserts that every bounded infinite sequence of points in a Euclidean space has a convergent subsequence.

The basis strategy for proving this theorem is exemplified in the following 3-dimensional example.

Example Let

$$(x_j, y_j, z_j) = \left(\sin(\pi\sqrt{j}),\ (-1)^j, \cos\left(\frac{2\pi \log j}{\log 2}\right)\right)$$

for $j = 1, 2, 3, \ldots$. This infinite sequence of points in $\mathbb{R}^3$ is bounded, because the components of its members all take values between -1 and 1. Moreover $x_j = 0$ whenever j is the square of a positive integer, $y_j = 1$ whenever j is even and $z_j = 1$ whenever j is a power of two.

The infinite sequence $x_1, x_2, x_3, \ldots$ has a convergent subsequence

$$x_1, x_4, x_9, x_{16}, x_{25}, \ldots$$

which includes those x_j for which j is the square of a positive integer. The corresponding subsequence $y_1, y_4, y_9, \ldots$ of $y_1, y_2, y_3, \ldots$ is not convergent, because its values alternate between 1 and -1. However this subsequence is bounded, and we can extract from this sequence a convergent subsequence

$$y_4, y_{16}, y_{36}, y_{64}, y_{100}, \ldots$$

which includes those x_j for which j is the square of an even positive integer.

The subsequence

$$x_4, x_{16}, x_{36}, y_{64}, y_{100}, \ldots$$

is also convergent, because it is a subsequence of a convergent subsequence. However the corresponding subsequence

$$z_4, z_{16}, z_{36}, z_{64}, z_{100}, \ldots$$

does not converge. (Indeed $z_j = 1$ when j is an even power of 2, but $z_j = \cos(2\pi \log(9)/\log(2))$ when $j = 9 \times 2^{2p}$ for some positive integer p.) However this subsequence is bounded, and we can extract from it a convergent subsequence

$$z_4, z_{16}, z_{64}, z_{256}, z_{1024}, \ldots$$

which includes those x_j for which j is equal to two raised to the power of an even positive integer. Then the first, second and third components of the following subsequence

$$(x_4, y_4, z_4), \;\; (x_{16}, y_{16}, z_{16}), \;\; (x_{64}, y_{64}, z_{64}), \;\; (x_{256}, y_{256}, z_{256}), \ldots$$

of the original sequence of points in $\mathbb{R}^3$ converge, and consequently this sequence is a convergent subsequence of the given infinite sequence of points in $\mathbb{R}^3$.

Example Let

$$x_j = \begin{cases} 1 & \text{if } j = 4k \text{ for some integer } k \\ 0 & \text{if } j = 4k+1 \text{ for some integer } k \\ -1 & \text{if } j = 4k+2 \text{ for some integer } k \\ 0 & \text{if } j = 4k+3 \text{ for some integer } k \end{cases}$$

and

$$y_j = \begin{cases} 0 & \text{if } j = 4k \text{ for some integer } k, \\ 1 & \text{if } j = 4k+1 \text{ for some integer } k, \\ 0 & \text{if } j = 4k+2 \text{ for some integer } k, \\ -1 & \text{if } j = 4k+3 \text{ for some integer } k, \end{cases},$$

and let $\mathbf{u}_j = (x_j, y_j)$ for $j = 1, 2, 3, 4, \ldots$. Then the first components x_j for which the index j is odd constitute a convergent sequence $x_1, x_3, x_5, x_7, \ldots$ of real numbers, and the second components y_j for which the index j is even also constitute a convergent sequence $y_2, y_4, y_6, y_8, \ldots$ of real numbers.

However one would not obtain a convergent subsequence of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots$ simply by selecting those indices j for which x_j is in the convergent subsequence $x_1, x_3, x_5, \ldots$ and y_j is in the convergent subsequence $y_2, y_4, y_6, \ldots$, because there no values of the index j for which x_j and y_j both belong to the respective subsequences. However the one-dimensional Bolzano-Weierstrass Theorem (Theorem **??**) guarantees that there is a convergent subsequence of $y_1, y_3, y_5, y_7, \ldots$, and indeed $y_1, y_5, y_9, y_{13}, \ldots$ is such a convergent subsequence. This yields a convergent subsequence $\mathbf{u}_1, \mathbf{u}_5, \mathbf{u}_9, \mathbf{u}_{13}, \ldots$ of the given bounded infinite sequence of points in $\mathbb{R}^2$.