MAU23203: Analysis in Several Real Variables
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Disquisition XII: An Example Concerning Second Order Partial Derivatives

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Let $f\colon X \to \mathbb{R}$ be a real-valued function on X. defined over an open subset X of $\mathbb{R}^n$. We consider the second order partial derivatives of the function f defined by

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_j}\right).$$

An important theorem establishes that if the first and second order partial derivatives

$$\frac{\partial f}{\partial x_i}, \quad \frac{\partial f}{\partial x_j}, \quad \frac{\partial^2 f}{\partial x_i \partial x_j} \text{ and } \frac{\partial^2 f}{\partial x_j \partial x_i}$$

all exist and are continuous then

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}.$$

In this disquisition, a counterexample is presented exhibiting a function f with the property that

$$\frac{\partial^2 f}{\partial x_i \partial x_j} \neq \frac{\partial^2 f}{\partial x_j \partial x_i}$$

at a particular point of the domain of the function at which the second order partial derivatives of the function fail to be continuous.

Example Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be the function defined by

$$f(x,y) = \begin{cases} \dfrac{xy(x^2-y^2)}{x^2+y^2} & \text{if } (x,y) \neq (0,0); \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

For convenience of notation, let us write

$$\begin{aligned} f_x(x,y) &= \frac{\partial f(x,y)}{\partial x}, \\ f_y(x,y) &= \frac{\partial f(x,y)}{\partial y}, \\ f_{xy}(x,y) &= \frac{\partial^2 f(x,y)}{\partial x \partial y}, \\ f_{yx}(x,y) &= \frac{\partial^2 f(x,y)}{\partial y \partial x}. \end{aligned}$$

If $(x,y) \neq (0,0)$ then

$$\begin{aligned} f_x &= \frac{\partial}{\partial x}\left(\frac{xy(x^2-y^2)}{x^2+y^2}\right) \\ &= \frac{(3x^2y-y^3)(x^2+y^2) - 2x^2y(x^2-y^2)}{(x^2+y^2)^2} \\ &= \frac{3x^4y + 3x^2y^3 - x^2y^3 - y^5 - 2x^4y + 2x^2y^3}{(x^2+y^2)^2} \\ &= \frac{x^4y + 4x^2y^3 - y^5}{(x^2+y^2)^2}. \end{aligned}$$

Thus

$$f_x = \frac{x^4y + 4x^2y^3 - y^5}{(x^2+y^2)^2}.$$

Similarly

$$f_y = \frac{-xy^4 - 4x^3y^2 + x^5}{(x^2+y^2)^2}.$$

(This can be deduced from the formula for f_x on noticing that $f(x,y)$ changes sign on interchanging the variables x and y.)

Differentiating again, when $(x,y) \neq (0,0)$, we find that

$$\begin{aligned} f_{xy}(x,y) &= \frac{\partial f_y}{\partial x} = \frac{\partial}{\partial x}\left(\frac{-xy^4 - 4x^3y^2 + x^5}{(x^2+y^2)^2}\right) \\ &= \frac{(-y^4 - 12x^2y^2 + 5x^4)(x^2+y^2)}{(x^2+y^2)^3} + \frac{-4x(-xy^4 - 4x^3y^2 + x^5)}{(x^2+y^2)^3} \\ &= \frac{-x^2y^4 - 12x^4y^2 + 5x^6 - y^6 - 12x^2y^4 + 5x^4y^2}{(x^2+y^2)^3} \end{aligned}$$

$$
\begin{aligned}
& + \frac{4x^2y^4 + 16x^4y^2 - 4x^6}{(x^2+y^2)^3} \\
= & \quad \frac{x^6 + 9x^4y^2 - 9x^2y^4 - y^6}{(x^2+y^2)^3}.
\end{aligned}
$$

Now the expression just obtained for f_{xy} when $(x, y) \neq (0, 0)$ changes sign when the variables x and y are interchanged. The same is true of the expression defining $f(x, y)$. It follows that f_{yx}. We conclude therefore that if $(x, y) \neq (0, 0)$ then

$$
f_{xy} = f_{yx} = \frac{x^6 + 9x^4y^2 - 9x^2y^4 - y^6}{(x^2+y^2)^3}.
$$

Now if $(x, y) \neq (0, 0)$ and if $r = \sqrt{x^2 + y^2}$ then

$$
\begin{aligned}
|f_x(x,y)| &= \left| \frac{x^4y + 4x^2y^3 - y^5}{(x^2+y^2)^2} \right| \\
&= \frac{|x^4y + 4x^2y^3 - y^5|}{r^4} \leq \frac{6r^5}{r^4} = 6r.
\end{aligned}
$$

It follows that

$$
\lim_{(x,y)\to(0,0)} f_x(x,y) = 0.
$$

Similarly

$$
\lim_{(x,y)\to(0,0)} f_y(x,y) = 0.
$$

However

$$
\lim_{(x,y)\to(0,0)} f_{xy}(x,y)
$$

does not exist. Now we have shown that

$$
f_{xy} = f_{yx} = \frac{x^6 + 9x^4y^2 - 9x^2y^4 - y^6}{(x^2+y^2)^3}.
$$

when $(x, y) \neq (0, 0)$. Consequently

$$
\begin{aligned}
\lim_{x\to 0} f_{xy}(x,0) = \lim_{x\to 0} f_{yx}(x,0) &= \lim_{x\to 0} \frac{x^6}{x^6} = 1, \\
\lim_{y\to 0} f_{xy}(0,y) = \lim_{y\to 0} f_{yx}(0,y) &= \lim_{y\to 0} \frac{-y^6}{y^6} = -1.
\end{aligned}
$$

Next we show that f_x, f_y, f_{xy} and f_{yx} all exist at $(0, 0)$, and thus exist everywhere on $\mathbb{R}^2$. Now the factor xy occurs in the numerator of the expression defining the value of $f(x,y)$ when $(x,y) \neq (0,0)$. Consequently $f(x,0) = 0$ for all real numbers x and $f(0,y) = 0$ for all real numbers y, and therefore $f_x(0,0) = 0$ and $f_y(0,0) = 0$. Also we previously found that

$$f_x = \frac{x^4 y + 4x^2 y^3 - y^5}{(x^2+y^2)^2} \quad \text{and} \quad f_y = \frac{-xy^4 - 4x^3 y^2 + x^5}{(x^2+y^2)^2}.$$

when x and y are not both equal to zero. Substituting $y = 0$ in the formula for f_y, and $x = 0$ for the formula for f_x, we find that

$$f_y(x,0) = x, \qquad f_x(0,y) = -y$$

for all $x, y \in \mathbb{R}$. We conclude that

$$\begin{aligned} f_{xy}(0,0) &= \left.\frac{d(f_y(x,0))}{dx}\right|_{x=0} = 1, \\ f_{yx}(0,0) &= \left.\frac{d(f_x(0,y))}{dy}\right|_{y=0} = -1, \end{aligned}$$

Thus

$$\frac{\partial^2 f}{\partial x \partial y} \neq \frac{\partial^2 f}{\partial y \partial x}$$

at $(0,0)$.

Observe that in this example the functions f_{xy} and f_{yx} are continuous throughout $\mathbb{R}^2 \setminus \{(0,0)\}$ and are equal to one another there. Although the functions f_{xy} and f_{yx} are well-defined at $(0,0)$, they are not continuous at $(0,0)$ and $f_{xy}(0,0) \neq f_{yx}(0,0)$.