Module MAU23203: Analysis in Several Real Variables
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Section 7: Fundamental Principles of Single Variable Calculus

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7 Fundamental Principles of Single Variable Calculus

7.1 Interior Points and Open Sets in the Real Line

Definition Let D be a subset of the set $\mathbb{R}$ of real numbers, and let s be a real number belonging to D. We say that s is an *interior point* of D if there exists some strictly positive number δ such that $x \in D$ for all real numbers x satisfying $s - \delta < x < s + \delta$. The *interior* of D is then the subset of D consisting of all real numbers belonging to D that are interior points of D.

It follows from the definition of open sets in the real line that a subset D of the set $\mathbb{R}$ of real numbers is an open set in $\mathbb{R}$ if and only if every point of D is an interior point of D.

Let s be a real number. We say that a function $f\colon D \to \mathbb{R}$ is defined *around* s if the real number s is an interior point of the domain D of the function f. It follows that the function f is defined around s if and only if there exists some strictly positive real number δ such that $f(x)$ is defined for all real numbers x satisfying $s - \delta < x < s + \delta$.

7.2 Differentiable Functions of a Single Real Variable

We recall some basic results of the theory of differentiable functions of a single real variable.

Definition Let s be some real number, and let f be a real-valued function defined around s. The function f is said to be *differentiable* at s, with *derivative* $f'(s)$, if and only if the limit

$$f'(s) = \lim_{h \to 0} \frac{f(s+h) - f(s)}{h}$$

is well-defined. We denote by f', or by $\dfrac{df}{dx}$ the function whose value at s is the derivative $f'(s)$ of f at s.

Let s be some real number, and let f and g be real-valued functions defined around s that are differentiable at s. The basic rules of differential calculus then ensure that the sum $f + g$ and the difference $f - g$ of the functions f and g are differentiable at s, and

$$(f + g)'(s) = f'(s) + g'(s), \qquad (f - g)'(s) = f'(s) - g'(s).$$

The product $f \cdot g$ of the differentiable functions f and g is also differentiable at s, and its derivative satisfies the identity

$$(f \cdot g)'(s) = f'(s)g(s) + f(s)g'(s) \quad (\textit{Product Rule}).$$

If moreover $g(s) \neq 0$ then the function f/g is differentiable at s (where $(f/g)(x) = f(x)/g(x)$ where both $f(x)$ and $g(x)$ are defined), and

$$(f/g)'(s) = \frac{f'(s)g(s) - f(s)g'(s)}{g(s)^2} \quad (\textit{Quotient Rule}).$$

Also if h is a real-valued function defined around $f(s)$ which is differentiable at $f(s)$ then the composition function $h \circ f$ is differentiable at $f(s)$ and

$$(h \circ f)'(s) = h'(f(s))f'(s) \quad (\textit{Chain Rule}).$$

Derivatives of some standard functions are as follows:—

$$\frac{d}{dx}(x^m) = mx^{m-1}, \quad \frac{d}{dx}(\sin x) = \cos x, \quad \frac{d}{dx}(\cos x) = -\sin x,$$

$$\frac{d}{dx}(\exp x) = \exp x, \quad \frac{d}{dx}(\log x) = \frac{1}{x} \ (x > 0).$$

7.3 The Product Rule

Proposition 7.1 (Product Rule) *Let s be some real number, and let f and g be differentiable real-valued functions defined around s. Let $f \cdot g$ denote the product function, defined so that $(f \cdot g)(x) = f(x)g(x)$ for all real numbers x for which both $f(x)$ and $g(x)$ are defined. Then the product function $f \cdot g$ is also differentiable at s, and*

$$(f \cdot g)'(s) = f'(s)g(s) + f(s)g'(s).$$

Proof Let h be a real number close enough to zero to ensure that the functions f and g are defined at $s + h$. Then

$$\begin{aligned} &\frac{f(s+h)g(s+h) - f(s)g(s)}{h} \\ &\qquad = \frac{f(s+h) - f(s)}{h} g(s+h) + f(s)\frac{g(s+h) - g(s)}{h}. \end{aligned}$$

Now $\lim_{h \to 0} g(s+h) = g(s)$ because the differentiable function g is necessarily continuous at s. Also limits of sums and products of functions are the sums and products of the respective limits where those limits are defined. It follows that

$$\begin{aligned}
&\lim_{h\to 0}\frac{f(s+h)g(s+h)-f(s)g(s)}{h}\\
&\qquad = \lim_{h\to 0}\frac{f(s+h)-f(s)}{h}\lim_{h\to 0}g(s+h)+f(s)\lim_{h\to 0}\frac{g(s+h)-g(s)}{h}\\
&\qquad = f'(s)g(s)+f(s)g'(s).
\end{aligned}$$

Thus the function $f\cdot g$ is differentiable at s, and

$$(f\cdot g)'(s)=f'(s)g(s)+f(s)g'(s),$$

as required. ∎

7.4 The Quotient Rule

Proposition 7.2 (Quotient Rule) *Let s be some real number, and let f and g be differentiable real-valued functions defined around s, where $g(s)\neq 0$. Let f/g denote the product function, defined so that $(f/g)(x)=f(x)/g(x)$ for all real numbers x for which $f(x)$ and $g(x)$ are defined and $g(x)\neq 0$. Then the quotient function f/g is differentiable at s, and*

$$(f/g)'(s)=\frac{f'(s)g(s)-f(s)g'(s)}{g(s)^2}.$$

Proof Let h be a non-zero real number that is close enough to zero to ensure that both $f(s+h)$ and $g(s+h)$ are defined and that $g(s+h)\neq 0$. Then

$$\begin{aligned}
\frac{f(s+h)}{g(s+h)}-\frac{f(s)}{g(s)} &= \frac{f(s+h)g(s)-f(s)g(s+h)}{g(s+h)g(s)}\\
&= \frac{(f(s+h)-f(s))\,g(s)-f(s)\,(g(s+h)-g(s))}{g(s)g(s+h)}.
\end{aligned}$$

Now $\lim_{h\to 0}g(s+h)=g(s)$ because the differentiable function g is necessarily continuous at s. Also limits of sums, products and quotients of functions are the sums, products and quotients of the respective limits where those limits and quotients are defined. It follows that

$$\begin{aligned}
&(f/g)'(s)\\
&\quad = \lim_{h\to 0}\frac{1}{h}\left(\frac{f(s+h)}{g(s+h)}-\frac{f(s)}{g(s)}\right)
\end{aligned}$$

$$
\begin{aligned}
&= \lim_{h\to 0}\left(\frac{1}{g(s+h)g(s)}\right) \\
&\quad \times \left(\lim_{h\to 0}\frac{f(s+h)-f(s)}{h}g(s) - f(s)\lim_{h\to 0}\frac{g(s+h)-g(s)}{h}\right) \\
&= \frac{f'(s)g(s)-f(s)g'(s)}{g(s)^2},
\end{aligned}
$$

as required. ∎

7.5 The Chain Rule

Proposition 7.3 (Chain Rule) *Let s be some real number, let f be a real-valued function defined around s, and let g be a real-valued function defined around $f(s)$. Suppose that the function f is differentiable at s, and the function g is differentiable at $f(s)$. Then the composition function $g \circ f$ is differentiable at s, and*

$$(g \circ f)'(s) = g'(f(s))f'(s).$$

Proof Let $r = f(s)$, and let

$$Q(y) = \begin{cases} \dfrac{g(y)-g(r)}{y-r} & \text{if } y \neq r; \\ g'(r) & \text{if } y = r. \end{cases}$$

for values of y around r. By considering separately the cases when $f(s+h) \neq f(s)$ and $f(s+h) = f(s)$, we see that

$$g(f(s+h)) - g(f(s)) = Q(f(s+h))(f(s+h)-f(s)).$$

Now the function Q is continuous at r, where $r = f(s)$, because

$$\lim_{y\to r} Q(y) = \lim_{y\to r}\frac{g(y)-g(r)}{y-r} = g'(r) = Q(r).$$

Also the function f is continuous at s, because it is differentiable at s. It follows that the composition function $Q \circ f$ is continuous at s, and thus

$$\lim_{h\to 0} Q(f(s+h)) = Q(f(s)) = g'(f(s)).$$

The limit of a product of functions is the product of the respective limits. Applying this result, we see that

$$
\begin{aligned}
(g \circ f)'(s) &= \lim_{h\to 0}\frac{g(f(s+h))-g(f(s))}{h} \\
&= \lim_{h\to 0} Q(f(s+h)) \lim_{h\to 0}\frac{f(s+h)-f(s)}{h} \\
&= g'(f(s))f'(s).
\end{aligned}
$$

The result follows. ∎

7.6 Rolle's Theorem

Theorem 7.4 (Rolle's Theorem) *Let $f\colon [a,b] \to \mathbb{R}$ be a real-valued function defined on some interval $[a,b]$, where $a < b$. Suppose that f is continuous on $[a,b]$ and is differentiable on (a,b). Suppose also that $f(a) = f(b)$. Then there exists some real number s satisfying $a < s < b$ which has the property that $f'(s) = 0$.*

Proof First we show that if the function f attains its minimum value at u, and if $a < u < b$, then $f'(u) = 0$. Now the difference quotient

$$\frac{f(u+h) - f(u)}{h}$$

is non-negative for all sufficiently small positive values of h; therefore $f'(u) \geq 0$. On the other hand, this difference quotient is non-positive for all sufficiently small negative values of h; therefore $f'(u) \leq 0$. We deduce therefore that $f'(u) = 0$.

Similarly if the function f attains its maximum value at v, and if $a < v < b$, then $f'(v) = 0$. (Indeed the result for local maxima can be deduced from the corresponding result for local minima simply by replacing the function f by $-f$.)

Now the function f is continuous on the closed bounded interval $[a,b]$. It therefore follows from the Extreme Value Theorem that there must exist real numbers u and v in the interval $[a,b]$ with the property that $f(u) \leq f(x) \leq f(v)$ for all real numbers x satisfying $a \leq x \leq b$. If $a < u < b$ then $f'(u) = 0$ and we can take $s = u$. Similarly if $a < v < b$ then $f'(v) = 0$ and we can take $s = v$.

The only remaining case to consider is when both u and v are endpoints of the interval $[a,b]$. In that case the function f is constant on $[a,b]$, since $f(a) = f(b)$, and we can choose s to be any real number satisfying $a < s < b$. ∎

7.7 The Mean Value Theorem

Rolle's Theorem can be generalized to yield the following important theorem.

Theorem 7.5 (The Mean Value Theorem) *Let $f\colon [a,b] \to \mathbb{R}$ be a real-valued function defined on some interval $[a,b]$. Suppose that f is continuous* *on $[a, b]$ and is differentiable on (a, b). Then there exists some real number s satisfying $a < s < b$ which has the property that*

$$f(b) - f(a) = f'(s)(b - a).$$

Proof Let $g\colon [a, b] \to \mathbb{R}$ be the real-valued function on the closed interval $[a, b]$ defined by

$$g(x) = f(x) - \frac{b - x}{b - a} f(a) - \frac{x - a}{b - a} f(b).$$

Then the function g is continuous on $[a, b]$ and differentiable on (a, b). Moreover $g(a) = 0$ and $g(b) = 0$. It follows from Rolle's Theorem (Theorem 7.4) that $g'(s) = 0$ for some real number s satisfying $a < s < b$. But

$$g'(s) = f'(s) - \frac{f(b) - f(a)}{b - a}.$$

Therefore $f(b) - f(a) = f'(s)(b - a)$, as required. ∎

A number of basic principles of single variable calculus follow as immediate consequences of the Mean Value Theorem (Theorem 7.5). A number of such consequences are presented in the following corollaries.

Corollary 7.6 *Let $f\colon [a, b] \to \mathbb{R}$ be a real-valued function defined on some interval $[a, b]$. Suppose that f is continuous on $[a, b]$ and is differentiable on (a, b) and that $f'(x) > 0$ for all real numbers x satisfying $a < x < b$. Then $f(b) > f(a)$.*

Corollary 7.7 *Let $f\colon [a, b] \to \mathbb{R}$ be a real-valued function defined on some interval $[a, b]$. Suppose that f is continuous on $[a, b]$ and is differentiable on (a, b) and that $f'(x) = 0$ for all real numbers x satisfying $a < x < b$. Then $f(x) = f(a)$ for all $x \in [a, b]$.*

Corollary 7.8 *Let $f\colon [a, b] \to \mathbb{R}$ be a real-valued function defined on some interval $[a, b]$, and let M be a real number. Suppose that f is continuous on $[a, b]$ and is differentiable on (a, b) and that $f'(x) \leq M$ for all real numbers x satisfying $a < x < b$. Then $f(x) \leq f(a) + M(x - a)$ for all $x \in [a, b]$.*

Corollary 7.9 *Let $f\colon [a, b] \to \mathbb{R}$ be a real-valued function defined on some interval $[a, b]$, and let M be a real number. Suppose that f is continuous on $[a, b]$ and is differentiable on (a, b) and that $|f'(x)| \leq M$ for all real numbers x satisfying $a < x < b$. Then $|f(b) - f(a)| \leq M(b - a)$.*

7.8 Concavity and the Second Derivative

Proposition 7.10 *Let s and h be real numbers, and let f be a twice differentiable real-valued function defined on some open interval containing s and $s+h$. Then there exists a real number θ satisfying $0 < \theta < 1$ for which*

$$f(s+h) = f(s) + hf'(s) + \tfrac{1}{2}h^2 f''(s+\theta h).$$

Proof Let I be an open interval, containing the real numbers 0 and 1, chosen to ensure that $f(s+th)$ is defined for all $t \in I$, and let $q: I \to \mathbb{R}$ be defined so that

$$q(t) = f(s+th) - f(s) - thf'(s) - t^2(f(s+h) - f(s) - hf'(s)).$$

for all $t \in I$. Differentiating, we find that

$$q'(t) = hf'(s+th) - hf'(s) - 2t(f(s+h) - f(s) - hf'(s))$$

and

$$q''(t) = h^2 f''(s+th) - 2(f(s+h) - f(s) - hf'(s)).$$

Now $q(0) = q(1) = 0$. It follows from Rolle's Theorem, applied to the function q on the interval $[0,1]$, that there exists some real number φ satisfying $0 < \varphi < 1$ for which $q'(\varphi) = 0$.

Then $q'(0) = q'(\varphi) = 0$, and therefore Rolle's Theorem can be applied to the function q' on the interval $[0, \varphi]$ to prove the existence of some real number θ satisfying $0 < \theta < \varphi$ for which $q''(\theta) = 0$. Then

$$0 = q''(\theta) = h^2 f''(s+\theta h) - 2(f(s+h) - f(s) - hf'(s)).$$

Rearranging, we find that

$$f(s+h) = f(s) + hf'(s) + \tfrac{1}{2}h^2 f''(s+\theta h),$$

as required. ∎

Corollary 7.11 *Let f be a twice-differentiable real-valued function defined throughout some open interval $(s-\delta_0, s+\delta_0)$ centred on a real number s. Suppose that $f''(s+h) > 0$ for all real numbers h satisfying $|h| < \delta_0$. Then*

$$f(s+h) \geq f(s) + hf'(s)$$

for all real numbers h satisfying $|h| < \delta_0$.

It follows from Corollary 7.11 that if a twice-differentiable function has positive second derivative throughout some open interval, then it is concave upwards throughout that interval. In particular the function has a local minimum at any point of that open interval where the first derivative is zero and the second derivative is positive.

Corollary 7.12 *Let $f\colon D \to \mathbb{R}$ be a twice-differentiable function defined over a subset D of $\mathbb{R}$, and let s be a point in the interior of D. Suppose that $f'(s) = 0$ and that $f''(x) > 0$ for all real numbers x belonging to some sufficiently small neighbourhood of x. Then s is a local minimum for the function f.*

7.9 Real-Analytic Functions

Definition Let f be a real-valued function defined over an open set in $\mathbb{R}$. The function f is said to be *real-analytic* if, given any real number s belonging to the domain of that function, the nth derivative $f^{(n)}(s)$ of the function s exists for all positive integers n and moreover there exists some positive real number δ with the property that

$$f(s+h) = f(s) + \sum_{n=1}^{+\infty} \frac{h^n}{n!} f^{(n)}(s).$$

for all real numbers h for which $|h| < \delta$.

The functions that one meets in basic mathematics, and in introductory calculus courses, are typically real-analytic. Examples of real-analytic functions include polynomial functions, the sine and cosine functions, the exponential function, and the natural logarithm function defined over the set of positive real numbers. Moreover sums, differences, products, quotients and compositions of real-analytic functions are real-analytic over the open sets over which the relevant sums, differences, products, quotients and compositions are defined.

However real-valued functions that have derivatives of all orders are not necessarily real-analytic. A standard counter-example is the real-valued function f on the real line defined such that

$$f(x) = \begin{cases} \exp\left(-\dfrac{1}{x^2}\right) & \text{if } x \neq 0; \\ 0 & \text{if } x = 0; \end{cases}$$

This function is very smooth around $x = 0$. Indeed one can show that $f^{(n)}(0) = 0$ for all positive integers n. Accordingly the value of function $f(h)$

for real numbers h close to zero cannot be represented around as the sum of the power series in h specified in the definition of a real-analytic function previously given.

Definition Let a real-valued function f with derivatives of all orders be defined over an open set in $\mathbb{R}$, and let s be some real number at which the function f is defined. Then the power series in the variable h that takes the form

$$f(s) + \sum_{n=1}^{+\infty} \frac{h^n}{n!} f^{(n)}(s)$$

is referred to as the *Taylor expansion* of the function f around s.

Accordingly a real-valued function with derivatives of all orders defined over an open subset of $\mathbb{R}$ is real-analytic if and only if the Taylor expansion of the function about any point of its domain converges to the function over some open interval containing that point.

7.10 Taylor's Theorem

The result obtained in Proposition 7.10 is a special case of a more general result. That more general result is a version of Taylor's Theorem with remainder. The proof of this theorem will make use of the following lemma.

Lemma 7.13 *Let s and h be real numbers, let f be a k times differentiable real-valued function defined on some open interval containing s and $s + h$, let $c_0, c_1, \ldots, c_{k-1}$ be real numbers, and let*

$$p(t) = f(s + th) - \sum_{n=0}^{k-1} c_n t^n.$$

for all real numbers t belonging to some open interval D for which $0 \in D$ and $1 \in D$. Then $p^{(n)}(0) = 0$ for all integers n satisfying $0 \leq n < k$ if and only if

$$c_n = \frac{h^n f^{(n)}(s)}{n!}$$

for all integers n satisfying $0 \leq n < k$.

Proof On setting $t = 0$, we find that $p(0) = f(s) - c_0$, and thus $p(0) = 0$ if and only if $c_0 = f(s)$.

Let the integer n satisfy $0 < n < k$. On differentiating $p(t)$ n times with respect to t, we find that

$$p^{(n)}(t) = h^n f^{(n)}(s + th) - \sum_{j=n}^{k-1} \frac{j!}{(j-n)!} c_j t^{j-n}.$$

Then, on setting $t = 0$, we find that only the term with $j = n$ contributes to the value of the sum on the right hand side of the above identity, and therefore

$$p^{(n)}(0) = h^n f^{(n)}(s) - n! c_n.$$

The result follows. ∎

Theorem 7.14 (Taylor's Theorem) *Let s and h be real numbers, and let f be a k times differentiable real-valued function defined on some open interval containing s and $s + h$. Then*

$$f(s + h) = f(s) + \sum_{n=1}^{k-1} \frac{h^n}{n!} f^{(n)}(s) + \frac{h^k}{k!} f^{(k)}(s + \theta h)$$

for some real number θ satisfying $0 < \theta < 1$.

Proof Let D be an open interval, containing the real numbers 0 and 1, chosen to ensure that $f(s + th)$ is defined for all $t \in D$, and let $p\colon D \to \mathbb{R}$ be defined so that

$$p(t) = f(s + th) - f(s) - \sum_{n=1}^{k-1} \frac{t^n h^n}{n!} f^{(n)}(s)$$

for all $t \in D$. A straightforward calculation shows that $p^{(n)}(0) = 0$ for $n = 0, 1, \ldots, k - 1$ (see Lemma 7.13). Thus if $q(t) = p(t) - p(1)t^k$ for all $s \in [0, 1]$ then $q^{(n)}(0) = 0$ for $n = 0, 1, \ldots, k - 1$, and $q(1) = 0$.

We can now apply Rolle's Theorem (Theorem 7.4) to the function q on the interval $[0, 1]$ to deduce the existence of some real number θ_1 satisfying $0 < \theta_1 < 1$ for which $q'(\theta_1) = 0$. We can then apply Rolle's Theorem to the function q' on the interval $[0, \theta_1]$ to deduce the existence of some real number θ_2 satisfying $0 < \theta_2 < \theta_1$ for which $q''(\theta_2) = 0$. By continuing in this fashion, applying Rolle's Theorem in turn to the functions $q'', q''', \ldots, q^{(k-1)}$, we deduce the existence of real numbers $\theta_1, \theta_2, \ldots, \theta_k$ satisfying $0 < \theta_k < \theta_{k-1} < \cdots < \theta_1 < 1$ with the property that $q^{(n)}(\theta_n) = 0$ for $n = 1, 2, \ldots, k$.

Let $\theta = \theta_k$. Then $0 < \theta < 1$ and

$$0 = \frac{1}{k!} q^{(k)}(\theta) = \frac{1}{k!} p^{(k)}(\theta) - p(1) = \frac{h^k}{k!} f^{(k)}(s + \theta h) - p(1).$$

It follows that

$$\begin{aligned} f(s+h) &= f(s) + \sum_{n=1}^{k-1} \frac{h^n}{n!} f^{(n)}(s) + p(1) \\ &= f(s) + \sum_{n=1}^{k-1} \frac{h^n}{n!} f^{(n)}(s) + \frac{h^k}{k!} f^{(k)}(s+\theta h), \end{aligned}$$

as required. ∎

Corollary 7.15 *Let $f\colon D \to \mathbb{R}$ be a k-times continuously differentiable function defined over an open subset D of $\mathbb{R}$ and let $s \in \mathbb{R}$. Then given any strictly positive real number ε, there exists some strictly positive real number δ such that*

$$\left| f(s+h) - f(s) - \sum_{n=1}^{k} \frac{h^n}{n!} f^{(n)}(s) \right| < \varepsilon |h|^k$$

whenever $|h| < \delta$.

Proof The function f is k-times continuously differentiable, and therefore its kth derivative $f^{(k)}$ is continuous. Let some strictly positive real number ε be given. It follows from the continuity of $f^{(k)}$ that there exists some strictly positive real number δ that is small enough to ensure that $s + h \in D$ and $|f^{(k)}(s+h) - f^{(k)}(s)| < k!\varepsilon$ whenever $|h| < \delta$. If h is an real number satisfying $|h| < \delta$, and if θ is a real number satisfying $0 < \theta < 1$, then $s + \theta h \in D$ and $|f^{(k)}(s + \theta h) - f^{(k)}(s)| < k!\varepsilon$. Now it follows from Taylor's Theorem (Theorem 7.14) that, given any real number h satisfying $|h| < \delta$ there exists some real number θ satisfying $0 < \theta < 1$ for which

$$f(s+h) = f(s) + \sum_{n=1}^{k-1} \frac{h^n}{n!} f^{(n)}(s) + \frac{h^k}{k!} f^{(k)}(s+\theta h).$$

Then

$$\begin{aligned} \left| f(s+h) - f(s) - \sum_{n=1}^{k} \frac{h^n}{n!} f^{(n)}(s) \right| &= \frac{|h|^k}{k!} |f^{(k)}(s+\theta h) - f^{(k)}(s)| \\ &< \varepsilon |h|^k, \end{aligned}$$

as required. ∎

7.11 Darboux Sums and the Riemann Integral

We now set out the basic definitions and state some basic results concerning the theory of integration of functions of a real variable that was developed by Jean-Gaston Darboux (1842–1917). The integral defined using lower and upper sums in the manner described below is sometimes referred to as the *Darboux integral* of a function on a given interval. However the class of functions that are integrable according to the definitions introduced by Darboux is the class of *Riemann-integrable* functions. Thus the approach using Darboux sums provides a convenient approach to define and establish the basic properties of the *Riemann integral*.

Definition A *partition* P of an interval $[a, b]$ is a set $\{u_0, u_1, u_2, \ldots, u_N\}$ of real numbers satisfying $a = u_0 < u_1 < u_2 < \cdots < u_{N-1} < u_N = b$.

Given any bounded real-valued function f on $[a, b]$, the *upper sum* (or *upper Darboux sum*) $U(P, f)$ of f for the partition P of $[a, b]$ is defined so that

$$U(P, f) = \sum_{i=1}^{N} M_i(u_i - u_{i-1}),$$

where $M_i = \sup\{f(x) : u_{i-1} \le x \le u_i\}$.

Similarly the *lower sum* (or *lower Darboux sum*) $L(P, f)$ of f for the partition P of $[a, b]$ is defined so that

$$L(P, f) = \sum_{i=1}^{N} m_i(u_i - u_{i-1}),$$

where $m_i = \inf\{f(x) : u_{i-1} \le x \le u_i\}$.

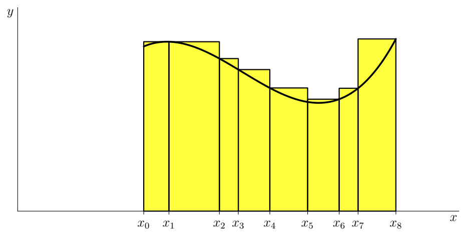
Clearly $L(P, f) \le U(P, f)$. Moreover $\sum\limits_{i=1}^{N}(u_i - u_{i-1}) = b - a$, and therefore

$$m(b - a) \le L(P, f) \le U(P, f) \le M(b - a),$$

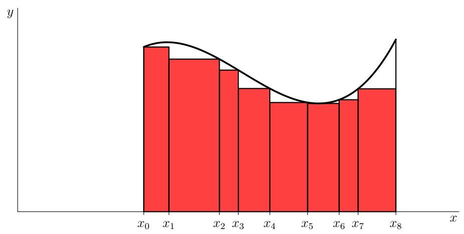
for any real numbers m and M satisfying $m \le f(x) \le M$ for all $x \in [a, b]$.

Definition Let f be a bounded real-valued function on the interval $[a, b]$, where $a < b$. The *upper Riemann integral* $\mathcal{U}\int_a^b f(x)\,dx$ (or *upper Darboux integral*) and the *lower Riemann integral* $\mathcal{L}\int_a^b f(x)\,dx$ (or *lower Darboux integral*) of the function f on $[a, b]$ are defined by

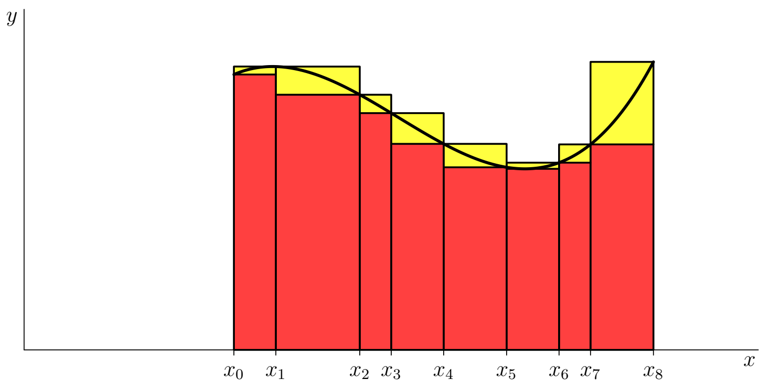
$$\begin{aligned} \mathcal{U}\int_a^b f(x)\,dx &= \inf\{U(P, f) : P \text{ is a partition of } [a, b]\}, \\ \mathcal{L}\int_a^b f(x)\,dx &= \sup\{L(P, f) : P \text{ is a partition of } [a, b]\}. \end{aligned}$$



y
x
x_0 x_1 x_2 x_3 x_4 x_5 x_6 x_7 x_8



y
x
x_0 x_1 x_2 x_3 x_4 x_5 x_6 x_7 x_8



y
x
x_0 x_1 x_2 x_3 x_4 x_5 x_6 x_7 x_8

The definition of upper and lower integrals thus requires that $\mathcal{U}\int_a^b f(x)\,dx$ be the infimum of the values of $U(P,f)$ and that $\mathcal{L}\int_a^b f(x)\,dx$ be the supremum of the values of $L(P,f)$ as P ranges over all possible partitions of the interval $[a,b]$.

Definition A bounded function $f\colon [a,b] \to \mathbb{R}$ on a closed bounded interval $[a,b]$ is said to be *Riemann-integrable* (or *Darboux-integrable*) on $[a,b]$ if

$$\mathcal{U}\int_a^b f(x)\,dx = \mathcal{L}\int_a^b f(x)\,dx,$$

in which case the *Riemann integral* $\int_a^b f(x)\,dx$ (or *Darboux integral*) of f on $[a,b]$ is defined to be the common value of $\mathcal{U}\int_a^b f(x)\,dx$ and $\mathcal{L}\int_a^b f(x)\,dx$.

When $a > b$ we define

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

for all Riemann-integrable functions f on $[b,a]$. We set $\int_a^b f(x)\,dx = 0$ when $b = a$.

We now state without proof several results that follow as consequences of the definition of the Riemann integral. The proofs of these results are straightforward applications of the basic principles and standard proof techniques of real analysis.

Proposition 7.16 *Let p, a and b be real numbers. Then*

$$\int_a^b p\,dx = p(b-a).$$

Proposition 7.17 *Let f and g be bounded Riemann-integrable functions on the interval $[a,b]$. Suppose that $f(x) \leq g(x)$ for all $x \in [a,b]$. Then*

$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx.$$

Proposition 7.18 *Let $f\colon [a,b] \to \mathbb{R}$ and $g\colon [a,b] \to \mathbb{R}$ be bounded Riemann-integrable functions on a closed bounded interval $[a,b]$, where a and b are real numbers satisfying $a \leq b$, and let p be a real number. Then the functions $f+g$ and pf are Riemann-integrable on $[a,b]$, and moreover*

$$\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx,$$

and

$$\int_a^b (pf(x))\,dx = p\int_a^b f(x)\,dx.$$

Proposition 7.19 *Let f be a bounded real-valued function on the interval $[a, c]$. Then the function f is Riemann-integrable on $[a, c]$ if and only if it is Riemann-integrable on both $[a, b]$ and $[b, c]$, in which case*

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx.$$

7.12 Integrability of Continuous functions

Theorem 7.20 *Let a and b be real numbers satisfying $a < b$. Then any continuous real-valued function on the interval $[a, b]$ is Riemann-integrable.*

Proof Let f be a continuous real-valued function on $[a, b]$. Then f is bounded above and below on the interval $[a, b]$, and moreover $f\colon [a, b] \to \mathbb{R}$ is uniformly continuous on $[a, b]$. (These results follow from Theorem 5.10 and Theorem 5.11.) Therefore there exists some strictly positive real number δ such that $|f(x) - f(y)| < \varepsilon$ whenever $x, y \in [a, b]$ satisfy $|x - y| < \delta$.

Choose a partition P of the interval $[a, b]$ such that each subinterval in the partition has length less than δ. Write $P = \{u_0, u_1, \ldots, u_N\}$, where $a = u_0 < u_1 < \cdots < u_N = b$. Now if $u_{i-1} \leq x \leq u_i$ then $|x - u_i| < \delta$, and hence $f(u_i) - \varepsilon < f(x) < f(u_i) + \varepsilon$. It follows that

$$f(u_i) - \varepsilon \leq m_i \leq M_i \leq f(u_i) + \varepsilon \qquad (i = 1, 2, \ldots, N),$$

where $m_i = \inf\{f(x) : u_{i-1} \leq x \leq u_i\}$ and $M_i = \sup\{f(x) : u_{i-1} \leq x \leq u_i\}$. Therefore

$$\begin{aligned} &\sum_{i=1}^N f(u_i)(u_i - u_{i-1}) - \varepsilon(b - a) \\ &\qquad \leq \; L(P, f) \leq U(P, f) \\ &\qquad \leq \; \sum_{i=1}^N f(u_i)(u_i - u_{i-1}) + \varepsilon(b - a), \end{aligned}$$

where $L(P, f)$ and $U(P, f)$ denote the lower and upper sums of the function f for the partition P.

We have now shown that

$$0 \leq \mathcal{U}\int_a^b f(x)\,dx - \mathcal{L}\int_a^b f(x)\,dx \leq U(P, f) - L(P, f) \leq 2\varepsilon(b - a).$$

But this inequality must be satisfied for any strictly positive real number ε. Therefore

$$\mathcal{U}\int_a^b f(x)\,dx = \mathcal{L}\int_a^b f(x)\,dx,$$

and thus the function f is Riemann-integrable on $[a, b]$. ∎

7.13 The Fundamental Theorem of Calculus

Let a and b be real numbers satisfying $a < b$. One can show that all continuous functions on the interval $[a, b]$ are Riemann-integrable. However the task of calculating the Riemann integral of a continuous function directly from the definition is difficult if not impossible for all but the simplest functions. Thus to calculate such integrals one makes use of the Fundamental Theorem of Calculus.

Theorem 7.21 (The Fundamental Theorem of Calculus) *Let f be a continuous real-valued function on the interval $[a, b]$, where $a < b$. Then*

$$\frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x)$$

for all x satisfying $a < x < b$.

Proof Let some strictly positive real number ε be given, and let ε_0 be a real number chosen so that $0 < \varepsilon_0 < \varepsilon$. (For example, one could choose $\varepsilon_0 = \frac{1}{2}\varepsilon$.) Now the function f is continuous at x, where $a < x < b$. It follows that there exists some strictly positive real number δ which is small enough to ensure that

$$f(x) - \varepsilon_0 \leq f(t) \leq f(x) + \varepsilon_0$$

for all $t \in [a, b]$ satisfying $x - \delta < t < x + \delta$. Let $F(s) = \int_a^s f(t)\,dt$ for all real numbers s satisfying $a \leq s \leq b$. Then

$$\begin{aligned} F(x+h) &= \int_a^{x+h} f(t)\,dt = \int_a^x f(t)\,dt + \int_x^{x+h} f(t)\,dt \\ &= F(x) + \int_x^{x+h} f(t)\,dt \end{aligned}$$

whenever $x + h \in [a, b]$. Also

$$\frac{1}{h}\int_x^{x+h} f(x)\,dt = \frac{f(x)}{h}\int_x^{x+h} dt = f(x),$$

because $f(x)$ is constant as t varies between x and $x + h$. It follows that

$$\frac{F(x+h) - F(x)}{h} - f(x) = \frac{1}{h}\int_x^{x+h} (f(t) - f(x))\,dt$$

whenever $x + h \in [a, b]$. But if $0 < |h| < \delta$ and $x + h \in [a, b]$ then

$$-\varepsilon_0 \leq f(t) - f(x) \leq \varepsilon_0$$

for all real numbers t belonging to the closed interval with endpoints x and $x + h$, and therefore

$$-\varepsilon_0 |h| \leq \int_x^{x+h} (f(t) - f(x))\, dt \leq \varepsilon_0 |h|.$$

It follows that

$$\left| \frac{F(x+h) - F(x)}{h} - f(x) \right| \leq \varepsilon_0 < \varepsilon$$

whenever $x + h \in [a, b]$ and $0 < |h| < \delta$. We conclude that

$$\frac{d}{dx}\left(\int_a^x f(t)\, dt \right) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h} = f(x),$$

as required. ∎

Let $f\colon [a, b] \to \mathbb{R}$ be a continuous function on a closed interval $[a, b]$. We say that f is *continuously differentiable* on $[a, b]$ if the derivative $f'(x)$ of f exists for all x satisfying $a < x < b$, the one-sided derivatives

$$\begin{aligned} f'(a) &= \lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h}, \\ f'(b) &= \lim_{h \to 0^+} \frac{f(b) - f(b-h)}{h} \end{aligned}$$

exist at the endpoints of $[a, b]$, and the function f' is continuous on $[a, b]$.

Lemma 7.22 *Let a and b be real numbers satisfying $a < b$, let $f\colon [a, b] \to \mathbb{R}$ be a continuous real-valued function on the closed interval $[a, b]$, and let $F(x) = \int_a^x f(t)\, dt$ for all $x \in [a, b]$. Then the one-sided derivatives of F at the endpoints of $[a, b]$ exist,*

$$\lim_{h \to 0^+} \frac{F(a+h) - F(a)}{h} = f(a)$$

and

$$\lim_{h \to 0^+} \frac{F(b) - F(b-h)}{h} = f(b).$$

Proof We adapt the arguments presented in the proof of the Fundamental Theorem of Calculus previously given. Let $F(s) = \int_a^s f(t)\, dt$ for all real numbers s satisfying $a \leq s \leq b$. Also let some positive real number ε be given, and let ε_0 be chosen so that $0 < \varepsilon_0 < \varepsilon$. Then there exists a real number δ satisfying $0 < \delta < b - a$ which is small enough to ensure that

$$f(a) - \varepsilon_0 \leq f(t) \leq f(a) + \varepsilon_0$$

for all real numbers t satisfying $a < t < a + \delta$ and

$$f(b) - \varepsilon_0 \leq f(t) \leq f(b) + \varepsilon_0$$

for all real numbers t satisfying $b - \delta < t < b$.

It follows, as in the proof of the Fundamental Theorem of Calculus (Theorem 7.21), that

$$\left| \frac{F(a+h) - F(a)}{h} - f(a) \right| \leq \varepsilon_0 < \varepsilon$$

and

$$\left| \frac{F(b) - F(b-h)}{h} - f(b) \right| \leq \varepsilon_0 < \varepsilon$$

whenever $0 < h < \delta$. The result follows. ∎

Proposition 7.23 *Let f be a continuously differentiable real-valued function on the interval $[a, b]$. Then*

$$\int_a^b \frac{df(x)}{dx}\, dx = f(b) - f(a)$$

Proof Define $g\colon [a, b] \to \mathbb{R}$ by

$$g(x) = f(x) - f(a) - \int_a^x \frac{df(t)}{dt}\, dt.$$

Then $g(a) = 0$, and

$$\frac{dg(x)}{dx} = \frac{df(x)}{dx} - \frac{d}{dx}\left(\int_a^x \frac{df(t)}{dt}\, dt \right) = 0$$

for all x satisfying $a < x < b$, by the Fundamental Theorem of Calculus. Thus the function g has zero derivative on the interval $[a, b]$. It follows that $g(b) = g(a) = 0$ (see Corollary 7.7). The result follows. ∎

Corollary 7.24 (Integration by Parts) *Let f and g be continuously differentiable real-valued functions on the interval $[a, b]$. Then*

$$\int_a^b f(x)\, \frac{dg(x)}{dx}\, dx = f(b)g(b) - f(a)g(a) - \int_a^b g(x)\, \frac{df(x)}{dx}\, dx.$$

Proof This result follows from Proposition 7.23 on integrating the identity

$$f(x)\frac{dg(x)}{dx} = \frac{d}{dx}\left(f(x)g(x)\right) - g(x)\, \frac{df(x)}{dx}.$$ ∎

Corollary 7.25 (Integration by Substitution) *Let $u\colon [a,b] \to \mathbb{R}$ be a continuously differentiable monotonically increasing function on the interval $[a,b]$, and let $c = u(a)$ and $d = u(b)$. Then*

$$\int_c^d f(x)\,dx = \int_a^b f(u(t))\frac{du(t)}{dt}\,dt.$$

for all continuous real-valued functions f on $[c,d]$.

Proof Let F and G be the functions on $[a,b]$ defined by

$$F(x) = \int_c^{u(x)} f(y)dy, \qquad G(x) = \int_a^x f(u(t))\frac{du(t)}{dt}dt.$$

Then $F(a) = 0 = G(a)$. Moreover $F(x) = H(u(x))$, where

$$H(s) = \int_c^s f(y)\,dy,$$

and $H'(s) = f(s)$ for all $s \in [a,b]$. Using the Chain Rule and the Fundamental Theorem of Calculus, we deduce that

$$F'(x) = H'(u(x))u'(x) = f(u(x))u'(x) = G'(x)$$

for all $x \in (a,b)$. The function $F - G$ has zero derivative on the interval $[a,b]$. It follows that $F(b) - G(b) = F(a) - G(a) = 0$ (see Corollary 7.7). Thus $H(d) = F(b) = G(b)$, which yields the required identity. ∎