Module MAU23203: Analysis in Several Real Variables
Michaelmas Term 2021
Section 6: Limits of Functions of Several Real Variables

D. R. Wilkins

Contents

6 Limits of Functions of Several Real Variables

6.1 Limit Points of Subsets of Euclidean Spaces

Definition Let X be a subset of n-dimensional Euclidean space $\mathbb{R}^n$, and let $\mathbf{p} \in \mathbb{R}^n$. The point $\mathbf{p}$ is said to be a *limit point* of the set X if, given any positive real number δ, there exists some point $\mathbf{x}$ of X for which $0 < |\mathbf{x} - \mathbf{p}| < \delta$.

6.2 Basic Properties of Limits of Functions of Several Real Variables

Definition Let X be a subset of m-dimensional Euclidean space $\mathbb{R}^m$, let $\varphi\colon X \to \mathbb{R}^n$ be a function mapping the set X into n-dimensional Euclidean space $\mathbb{R}^n$, let $\mathbf{p}$ be a limit point of the set X, and let $\mathbf{v}$ be a vector in $\mathbb{R}^n$. The point $\mathbf{v}$ is said to be the *limit* of $\varphi(\mathbf{x})$, as $\mathbf{x}$ tends to $\mathbf{p}$ in X, if and only if, given any strictly positive real number ε, there exists some strictly positive real number δ such that $|\varphi(\mathbf{x}) - \mathbf{v}| < \varepsilon$ whenever $\mathbf{x} \in X$ satisfies $0 < |\mathbf{x} - \mathbf{p}| < \delta$.

Let X be a subset of m-dimensional Euclidean space $\mathbb{R}^m$, let $\varphi\colon X \to \mathbb{R}^n$ be a function mapping the set X into n-dimensional Euclidean space $\mathbb{R}^n$, let $\mathbf{p}$ be a limit point of the set X, and let $\mathbf{v}$ be a vector in $\mathbb{R}^n$. If $\mathbf{v}$ is the limit of $\varphi(\mathbf{x})$ as $\mathbf{x}$ tends to $\mathbf{p}$ in X then we can denote this fact by writing $\lim_{\mathbf{x} \to \mathbf{p}} \varphi(\mathbf{x}) = \mathbf{v}$.

Proposition 6.1 *Let X be a subset of $\mathbb{R}^m$, let $\mathbf{p}$ be a limit point of X, and let $\mathbf{v}$ be a vector in $\mathbb{R}^n$. A function $\varphi\colon X \to \mathbb{R}^n$ has the property that*

$$\lim_{\mathbf{x} \to \mathbf{p}} \varphi(\mathbf{x}) = \mathbf{v}$$

if and only if

$$\lim_{\mathbf{x} \to \mathbf{p}} f_i(\mathbf{x}) = v_i$$

for $i = 1, 2, \ldots, n$, where $f_1, f_2, \ldots, f_n$ are the components of the function φ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$.

Proof Suppose that $\lim_{\mathbf{x} \to \mathbf{p}} \varphi(\mathbf{x}) = \mathbf{v}$. Let i be an integer between 1 and n, and let some positive real number ε be given. Then there exists some positive real number δ such that $|\varphi(\mathbf{x}) - \mathbf{v}| < \varepsilon$ whenever $0 < |\mathbf{x} - \mathbf{p}| < \delta$. It then follows from the definition of the Euclidean norm that

$$|f_i(\mathbf{x}) - v_i| \leq |\varphi(\mathbf{x}) - \mathbf{v}| < \varepsilon$$

whenever $0 < |\mathbf{x} - \mathbf{p}| < \delta$. Thus if $\lim_{\mathbf{x}\to\mathbf{p}} \varphi(\mathbf{x}) = \mathbf{v}$ then $\lim_{\mathbf{x}\to\mathbf{p}} f_i(\mathbf{x}) = v_i$ for $i = 1, 2, \ldots, n$.

Conversely suppose that

$$\lim_{\mathbf{x}\to\mathbf{p}} f_i(\mathbf{x}) = v_i$$

for $i = 1, 2, \ldots, n$. Let some positive real number ε be given. Then there exist positive real numbers $\delta_1, \delta_2, \ldots, \delta_n$ such that $|f_i(\mathbf{x}) - v_i| < \varepsilon/\sqrt{n}$ for $\mathbf{x} \in X$ satisfying $0 < |\mathbf{x} - \mathbf{p}| < \delta_i$. Let δ be the minimum of $\delta_1, \delta_2, \ldots, \delta_n$. If $\mathbf{x} \in X$ satisfies $0 < |\mathbf{x} - \mathbf{p}| < \delta$ then

$$|\varphi(\mathbf{x}) - \mathbf{v}|^2 = \sum_{i=1}^{n} (f_i(\mathbf{x}) - v_i)^2 < \varepsilon^2,$$

and hence $|\varphi(\mathbf{x}) - \mathbf{v}| < \varepsilon$. Thus

$$\lim_{\mathbf{x}\to\mathbf{p}} \varphi(\mathbf{x}) = \mathbf{v},$$

as required. ■

Proposition 6.2 *Let X be a subset of m-dimensional Euclidean space $\mathbb{R}^m$, let $\varphi: X \to \mathbb{R}^n$ and $\psi: X \to \mathbb{R}^n$ be functions mapping X into n-dimensional Euclidean space $\mathbb{R}^n$, let $\mathbf{p}$ be a limit point of X, and let $\mathbf{v}$ and $\mathbf{w}$ be points of $\mathbb{R}^n$. Suppose that*

$$\lim_{\mathbf{x}\to\mathbf{p}} \varphi(\mathbf{x}) = \mathbf{v}$$

and

$$\lim_{\mathbf{x}\to\mathbf{p}} \psi(\mathbf{x}) = \mathbf{w}.$$

Then

$$\lim_{\mathbf{x}\to\mathbf{p}} (\varphi(\mathbf{x}) + \psi(\mathbf{x})) = \mathbf{v} + \mathbf{w}.$$

Proof Let some strictly positive real number ε be given. Then there exist strictly positive real numbers δ_1 and δ_2 such that

$$|\varphi(\mathbf{x}) - \mathbf{v}| < \tfrac{1}{2}\varepsilon$$

whenever $\mathbf{x} \in X$ satisfies $0 < |\mathbf{x} - \mathbf{p}| < \delta_1$ and

$$|\psi(\mathbf{x}) - \mathbf{w}| < \tfrac{1}{2}\varepsilon$$

whenever $\mathbf{x} \in X$ satisfies $0 < |\mathbf{x} - \mathbf{p}| < \delta_2$. Let δ be the minimum of δ_1 and δ_2. Then $\delta > 0$, and if $\mathbf{x} \in X$ satisfies $0 < |\mathbf{x} - \mathbf{p}| < \delta$ then

$$|\varphi(\mathbf{x}) - \mathbf{v}| < \tfrac{1}{2}\varepsilon$$

and

$$|\psi(\mathbf{x}) - \mathbf{w}| < \tfrac{1}{2}\varepsilon,$$

and therefore

$$\begin{aligned} |\varphi(\mathbf{x}) + \psi(\mathbf{x}) - (\mathbf{v} + \mathbf{w})| &\leq |\varphi(\mathbf{x}) - \mathbf{v}| + |\psi(\mathbf{x}) - \mathbf{w}| \\ &< \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon. \end{aligned}$$

It follows that

$$\lim_{\mathbf{x} \to \mathbf{p}} (\varphi(\mathbf{x}) + \psi(\mathbf{x})) = \mathbf{v} + \mathbf{w},$$

as required. ∎

Lemma 6.3 *Let X and Y be subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively, let $\mathbf{p}$ be a limit point of X, let $\mathbf{v}$ be a point of Y, let $\varphi: X \to Y$ be a function mapping the set X into the set Y, and let $\psi: Y \to \mathbb{R}^k$ be a function mapping the set Y into $\mathbb{R}^k$. Suppose that*

$$\lim_{\mathbf{x} \to \mathbf{p}} \varphi(\mathbf{x}) = \mathbf{v}$$

and that the function ψ is continuous at $\mathbf{v}$. Then

$$\lim_{\mathbf{x} \to \mathbf{p}} \psi(\varphi(\mathbf{x})) = \psi(\mathbf{v}).$$

Proof Let some positive real number ε be given. Then there exists some positive real number η such that $|\psi(\mathbf{y}) - \psi(\mathbf{v})| < \varepsilon$ for all $\mathbf{y} \in Y$ satisfying $|\mathbf{y} - \mathbf{v}| < \eta$, because the function g is continuous at $\mathbf{v}$. But then there exists some positive real number δ such that $|\varphi(\mathbf{x}) - \mathbf{v}| < \eta$ for all $\mathbf{x} \in X$ satisfying $0 < |\mathbf{x} - \mathbf{p}| < \delta$. It follows that $|\psi(\varphi(\mathbf{x})) - \psi(\mathbf{v})| < \varepsilon$ for all $\mathbf{x} \in X$ satisfying $0 < |\mathbf{x} - \mathbf{p}| < \delta$, and thus

$$\lim_{\mathbf{x} \to \mathbf{p}} \psi(\varphi(\mathbf{x})) = \psi(\mathbf{v}),$$

as required. ∎

Proposition 6.4 *Let X be a subset of $\mathbb{R}^m$, let $f\colon X \to \mathbb{R}$ and $g\colon X \to \mathbb{R}$ be real-valued functions on X, and let $\mathbf{p}$ be a limit point of the set X. Suppose that $\lim_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x})$ and $\lim_{\mathbf{x}\to\mathbf{p}} g(\mathbf{x})$ both exist. Then so do $\lim_{\mathbf{x}\to\mathbf{p}}(f(\mathbf{x}) + g(\mathbf{x}))$, $\lim_{\mathbf{x}\to\mathbf{p}}(f(\mathbf{x}) - g(\mathbf{x}))$ and $\lim_{\mathbf{x}\to\mathbf{p}}(f(\mathbf{x})g(\mathbf{x}))$, and moreover*

$$\begin{aligned}
\lim_{\mathbf{x}\to\mathbf{p}}(f(\mathbf{x}) + g(\mathbf{x})) &= \lim_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x}) + \lim_{\mathbf{x}\to\mathbf{p}} g(\mathbf{x}),\\
\lim_{\mathbf{x}\to\mathbf{p}}(f(\mathbf{x}) - g(\mathbf{x})) &= \lim_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x}) - \lim_{\mathbf{x}\to\mathbf{p}} g(\mathbf{x}),\\
\lim_{\mathbf{x}\to\mathbf{p}}(f(\mathbf{x})g(\mathbf{x})) &= \lim_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x}) \times \lim_{\mathbf{x}\to\mathbf{p}} g(\mathbf{x}),
\end{aligned}$$

If moreover $g(\mathbf{x}) \neq 0$ for all $\mathbf{x} \in X$ and $\lim_{\mathbf{x}\to\mathbf{p}} g(\mathbf{x}) \neq 0$ then

$$\lim_{\mathbf{x}\to\mathbf{p}} \frac{f(\mathbf{x})}{g(\mathbf{x})} = \frac{\lim_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x})}{\lim_{\mathbf{x}\to\mathbf{p}} g(\mathbf{x})}.$$

Proof Let $q = \lim_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x})$ and $r = \lim_{\mathbf{x}\to\mathbf{p}} g(\mathbf{x})$, and let $\psi\colon X \to \mathbb{R}^2$ be defined such that

$$\psi(\mathbf{x}) = (f(\mathbf{x}), g(\mathbf{x}))$$

for all $\mathbf{x} \in X$. Then

$$\lim_{\mathbf{x}\to\mathbf{p}} \psi(\mathbf{x}) = (q, r)$$

(see Proposition 6.1).

Let $s\colon \mathbb{R}^2 \to \mathbb{R}$ and $m\colon \mathbb{R}^2 \to \mathbb{R}$ be the functions from $\mathbb{R}^2$ to $\mathbb{R}$ defined such that $s(u, v) = u + v$ and $m(u, v) = uv$ for all $u, v \in \mathbb{R}$. Then the functions s and m are continuous (see Lemma 5.4). Also $f + g = s \circ \psi$ and $f \cdot g = m \circ \psi$. It follows from this that

$$\begin{aligned}
\lim_{\mathbf{x}\to\mathbf{p}}(f(\mathbf{x}) + g(\mathbf{x})) &= \lim_{\mathbf{x}\to\mathbf{p}} s(f(\mathbf{x}), g(\mathbf{x})) = \lim_{\mathbf{x}\to\mathbf{p}} s(\psi(\mathbf{x}))\\
&= s\left(\lim_{\mathbf{x}\to\mathbf{p}} \psi(\mathbf{x})\right) = s(q, r) = q + r,
\end{aligned}$$

(see Lemma 6.3), and

$$\lim_{\mathbf{x}\to\mathbf{p}}(-g(\mathbf{x})) = -r.$$

It follows that

$$\lim_{\mathbf{x}\to\mathbf{p}}(f(\mathbf{x}) - g(\mathbf{x})) = q - r.$$

Similarly, when taking limits of products of functions,

$$\begin{aligned}\lim_{\mathbf{x}\to\mathbf{p}} (f(\mathbf{x})g(\mathbf{x})) &= \lim_{\mathbf{x}\to\mathbf{p}} m(f(\mathbf{x}), g(\mathbf{x})) = \lim_{\mathbf{x}\to\mathbf{p}} m(\psi(\mathbf{x})) \\ &= m\left(\lim_{\mathbf{x}\to\mathbf{p}} \psi(\mathbf{x})\right) = m(q, r) = qr\end{aligned}$$

Now suppose that $g(\mathbf{x}) \neq 0$ for all $\mathbf{x} \in X$ and that $\lim_{\mathbf{x}\to\mathbf{p}} g(\mathbf{x}) \neq 0$. Representing the function sending $\mathbf{x} \in X$ to $1/g(\mathbf{x})$ as the composition of the function g and the reciprocal function $e\colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$, where $e(t) = 1/t$ for all non-zero real numbers t, we find, as in the first proof, that the function sending each point $\mathbf{x}$ of X to

$$\lim_{\mathbf{x}\to\mathbf{p}} \left(\frac{1}{g(\mathbf{x})}\right) = \frac{1}{r}.$$

It then follows that

$$\lim_{\mathbf{x}\to\mathbf{p}} \frac{f(\mathbf{x})}{g(\mathbf{x})} = \frac{q}{r},$$

as required. ∎

6.3 Relationships between Limits and Continuity

Proposition 6.5 *Let X be a subset of $\mathbb{R}^m$, let $f\colon X \to \mathbb{R}^n$ be a function mapping the set X into $\mathbb{R}^n$, and let $\mathbf{p}$ be a point of the set X that is also a limit point of X. Then the function f is continuous at the point $\mathbf{p}$ if and only if $\lim_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x}) = f(\mathbf{p})$.*

Proof The result follows directly on comparing the relevant definitions. ∎

Let X be a subset of m-dimensional Euclidean space $\mathbb{R}^m$, and let $\mathbf{p}$ be a point of the set X. Suppose that the point $\mathbf{p}$ is not a limit point of the set X. Then there exists some strictly positive real number δ_0 such that $|\mathbf{x}-\mathbf{p}| \geq \delta_0$ for all $\mathbf{x} \in X$ satisfying $\mathbf{x} \neq \mathbf{p}$. The point $\mathbf{p}$ is then said to be an *isolated point* of X.

Let X be a subset of m-dimensional Euclidean space $\mathbb{R}^m$. The definition of continuity then ensures that any function $\varphi\colon X \to \mathbb{R}^n$ mapping the set X into n-dimensional Euclidean space $\mathbb{R}^n$ is continuous at any isolated point of its domain X.