Module MAU23203: Analysis in Several Real Variables
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Section 3: Convergence in Euclidean Spaces

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3 Convergence in Euclidean Spaces

3.1 Convergence of Infinite Sequences of Real Numbers

An *infinite sequence* $x_1, x_2, x_3, \ldots$ of real numbers associates to each positive integer j a corresponding real number x_j.

Definition An infinite sequence $x_1, x_2, x_3, \ldots$ of real numbers is said to *converge* to some real number p if and only if the following criterion is satisfied:

> given any strictly positive real number ε, there exists some positive integer N such that $|x_j - p| < \varepsilon$ for all positive integers j satisfying $j \geq N$.

If an infinite sequence $x_1, x_2, x_3, \ldots$ of real numbers converges to some real number p, then p is said to be the *limit* of the sequence, and we can indicate the convergence of the infinite sequence to p by writing '$x_j \to p$ as $j \to +\infty$', or by writing '$\lim_{j \to +\infty} x_j = p$'.

Let x and p be real numbers, and let ε be a strictly positive real number. Then $|x - p| < \varepsilon$ if and only if both $x - p < \varepsilon$ and $p - x < \varepsilon$. It follows that $|x - p| < \varepsilon$ if and only if $p - \varepsilon < x < p + \varepsilon$. The condition $|x - p| < \varepsilon$ essentially requires that the value of the real number x should agree with p to within an error of at most ε. An infinite sequence $x_1, x_2, x_3, \ldots$ of real numbers converges to some real number p if and only if, given any positive real number ε, there exists some positive integer N such that $p - \varepsilon < x_j < p + \varepsilon$ for all positive integers j satisfying $j \geq N$.

Definition We say that an infinite sequence $x_1, x_2, x_3, \ldots$ of real numbers is *bounded above* if there exists some real number B such that $x_j \leq B$ for all positive integers j. Similarly we say that this sequence is *bounded below* if there exists some real number A such that $x_j \geq A$ for all positive integers j. A sequence is said to be *bounded* if it is bounded above and bounded below. Thus the sequence $x_1, x_2, x_3, \ldots$ is bounded if and only if there exist real numbers A and B such that $A \leq x_j \leq B$ for all positive integers j.

Lemma 3.1 *Every convergent infinite sequence of real numbers is bounded.*

Proof Let $x_1, x_2, x_3, \ldots$ be an infinite sequence of real numbers that converges to some real number p. On applying the formal definition of convergence (with $\varepsilon = 1$), we deduce the existence of some positive integer N such that $p - 1 < x_j < p + 1$ for all $j \geq N$. But then $A \leq x_j \leq B$ for all positive integers j, where A is the minimum of $x_1, x_2, \ldots, x_{N-1}$ and $p - 1$, and B is the maximum of $x_1, x_2, \ldots, x_{N-1}$ and $p + 1$. ∎

3.2 Monotonic Sequences

An infinite sequence $x_1, x_2, x_3, \ldots$ of real numbers is said to be *strictly increasing* if $x_{j+1} > x_j$ for all positive integers j, *strictly decreasing* if $x_{j+1} < x_j$ for all positive integers j, *non-decreasing* if $x_{j+1} \geq x_j$ for all positive integers j, *non-increasing* if $x_{j+1} \leq x_j$ for all positive integers j. A sequence satisfying any one of these conditions is said to be *monotonic*; thus a monotonic sequence is either non-decreasing or non-increasing.

Theorem 3.2 *Any non-decreasing infinite sequence of real numbers that is bounded above is convergent. Similarly any non-increasing infinite sequence of real numbers that is bounded below is convergent.*

Proof Let $x_1, x_2, x_3, \ldots$ be a non-decreasing sequence of real numbers that is bounded above. It follows from the Least Upper Bound Principle that there exists a least upper bound p for the set $\{x_j : j \in \mathbb{N}\}$. We claim that the sequence converges to p.

Let some strictly positive real number ε be given. We must show that there exists some positive integer N such that $|x_j - p| < \varepsilon$ whenever $j \geq N$. Now $p - \varepsilon$ is not an upper bound for the set $\{x_j : j \in \mathbb{N}\}$ (because p is the least upper bound), and therefore there must exist some positive integer N such that $x_N > p - \varepsilon$. But then $p - \varepsilon < x_j \leq p$ whenever $j \geq N$, since the sequence is non-decreasing and bounded above by the real number p. Thus $|x_j - p| < \varepsilon$ whenever $j \geq N$. Therefore $x_j \to p$ as $j \to +\infty$, as required.

Next we note that if an infinite sequence $x_1, x_2, x_3, \ldots$ is non-increasing and bounded below then the sequence $-x_1, -x_2, -x_3, \ldots$ is non-decreasing and bounded above, and is therefore convergent. It follows that the sequence $x_1, x_2, x_3, \ldots$ is also convergent. ∎

3.3 Subsequences of Sequences of Real Numbers

Definition Let $x_1, x_2, x_3, \ldots$ be an infinite sequence of real numbers. A *subsequence* of this infinite sequence is a sequence of the form $x_{j_1}, x_{j_2}, x_{j_3}, \ldots$ where $j_1, j_2, j_3, \ldots$ is an infinite sequence of positive integers with

$$j_1 < j_2 < j_3 < \cdots.$$

Let $x_1, x_2, x_3, \ldots$ be an infinite sequence of real numbers. The following sequences are examples of subsequences of this sequence:—

$$x_1, x_3, x_5, x_7, \ldots$$
$$x_1, x_4, x_9, x_{16}, \ldots$$

3.4 The Bolzano-Weierstrass Theorem in One Dimension

Theorem 3.3 (Bolzano-Weierstrass for the Real Line) *Every bounded infinite sequence of real numbers has a convergent subsequence.*

Proof Let some bounded infinite sequence $x_1, x_2, x_3, \ldots$ of real numbers be given. We define a *peak index* to be a positive integer j with the property that $x_j \geq x_k$ for all positive integers k satisfying $k \geq j$. Thus a positive integer j is a peak index if and only if the jth member of the infinite sequence $x_1, x_2, x_3, \ldots$ is greater than or equal to all succeeding members of the sequence. Let S be the set consisting of all peak indices. Then

$$S = \{j \in \mathbb{N} : x_j \geq x_k \text{ for all } k \geq j\}.$$

First let us suppose that the set of peak indices is infinite. Arrange the set S of peak indices in increasing order so that $S = \{j_1, j_2, j_3, j_4, \ldots\}$, where $j_1 < j_2 < j_3 < j_4 < \cdots$. It follows from the definition of peak indices that $x_{j_1} \geq x_{j_2} \geq x_{j_3} \geq x_{j_4} \geq \cdots$. Thus $x_{j_1}, x_{j_2}, x_{j_3}, \ldots$ is a non-increasing subsequence of the given infinite sequence $x_1, x_2, x_3, \ldots$. This subsequence is bounded below (since the given infinite sequence is bounded). It follows from Theorem 3.2 that $x_{j_1}, x_{j_2}, x_{j_3}, \ldots$ is a convergent subsequence of the given infinite sequence.

Now suppose that the set S of peak indices is finite. Choose a positive integer j_1 which is greater than every peak index. Then j_1 is not a peak index. Therefore there must exist some positive integer j_2 satisfying $j_2 > j_1$ such that $x_{j_2} > x_{j_1}$. Moreover j_2 is not a peak index (because j_2 is greater than j_1 and j_1 in turn is greater than every peak index). Therefore there must exist some positive integer j_3 satisfying $j_3 > j_2$ such that $x_{j_3} > x_{j_2}$. We can continue in this fashion to construct (by induction on j) a strictly increasing subsequence $x_{j_1}, x_{j_2}, x_{j_3}, \ldots$ of our original sequence. This increasing subsequence is bounded above (since the original sequence is bounded) and thus is convergent, by Theorem 3.2. This completes the proof of the one-dimensional case of the Bolzano-Weierstrass Theorem. ∎

3.5 Convergence of Sequences in Euclidean Spaces

Definition An infinite sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ of points in $\mathbb{R}^n$ is said to *converge* to a point $\mathbf{p}$ if and only if, given strictly positive real number ε, there exists some positive integer N such that $|\mathbf{x}_j - \mathbf{p}| < \varepsilon$ whenever $j \geq N$.

Given a convergent infinite sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ of points in $\mathbb{R}^n$, the point $\mathbf{p}$ to which the sequence converges is referred to as the *limit* of the infinite sequence, and may be denoted by $\lim_{j\to+\infty} \mathbf{x}_j$.

Lemma 3.4 *Let $\mathbf{p}$ be a point of $\mathbb{R}^n$, where $\mathbf{p} = (p_1, p_2, \ldots, p_n)$. Then an infinite sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ of points in $\mathbb{R}^n$ converges to $\mathbf{p}$ if and only if the ith components of the elements of this sequence converge to p_i for $i = 1, 2, \ldots, n$.*

Proof For each positive integer j, let $(\mathbf{x}_j)_i$ denote the ith component of $\mathbf{x}_j$. Then $|(\mathbf{x}_j)_i - p_i| \leq |\mathbf{x}_j - \mathbf{p}|$ for $i = 1, 2, \ldots, n$ and for all positive integers j. It follows directly from the definition of convergence that if $\mathbf{x}_j \to \mathbf{p}$ as $j \to +\infty$ then $(\mathbf{x}_j)_i \to p_i$ as $j \to +\infty$.

Conversely suppose that, for each integer i between 1 and n, $(\mathbf{x}_j)_i \to p_i$ as $j \to +\infty$. Let some positive real number ε be given. Then there exist positive integers $N_1, N_2, \ldots, N_n$ such that $|(\mathbf{x}_j)_i - p_i| < \varepsilon/\sqrt{n}$ whenever $j \geq N_i$. Let N be the maximum of $N_1, N_2, \ldots, N_n$. If $j \geq N$ then $j \geq N_i$ for $i = 1, 2, \ldots, n$, and therefore

$$|\mathbf{x}_j - \mathbf{p}|^2 = \sum_{i=1}^{n} ((\mathbf{x}_j)_i - p_i)^2 < n \left(\frac{\varepsilon}{\sqrt{n}}\right)^2 = \varepsilon^2.$$

Thus $\mathbf{x}_j \to \mathbf{p}$ as $j \to +\infty$, as required. ∎

3.6 The Multidimensional Bolzano-Weierstrass Theorem

Theorem 3.5 (Multidimensional Bolzano-Weierstrass Theorem) *Every bounded sequence of points in a Euclidean space has a convergent subsequence.*

Proof The theorem is proved by induction on the dimension n of the space $\mathbb{R}^n$ within which the points reside. When $n = 1$, the required result is the one-dimensional case of the Bolzano-Weierstrass Theorem, and the result has already been established in this case (see Theorem 3.3).

When $n > 1$, the result is proved in dimension n asssuming the result in dimensions $n - 1$ and 1. Consequently the result is established successively in dimensions $2, 3, 4, \ldots$, and therefore is valid for bounded sequences in $\mathbb{R}^n$ for all positive integers n.

It has been shown that every bounded infinite sequence of real numbers has a convergent subsequence (Theorem 3.3). Let n be an integer greater than one, and suppose, as an induction hypothesis, that, in cases where $n > 2$, all bounded sequences of points in $\mathbb{R}^{n-1}$ have convergent subsequences. Let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ be a bounded infinite sequence in $\mathbf{R}^n$ and, for each positive integer j, let $\mathbf{s}_j$ denote the point of $\mathbb{R}^{n-1}$ whose ith component is equal to the ith component $x_{j,i}$ of $\mathbf{x}_j$ for each integer i between 1 and $n-1$.

Let some strictly positive real number ε be given. Now the infinite sequence

$$\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3, \ldots$$

of points of $\mathbb{R}^{n-1}$ is a bounded infinite sequence. In the case when $n = 2$ we can apply the one-dimensional Bolzano-Weierstrass Theorem (Theorem 3.3) to conclude that this sequence of real numbers has a convergent subsequence. In cases where $n > 2$, we are supposing as our induction hypothesis that any bounded sequence in $\mathbb{R}^{n-1}$ has a convergent subsequence. Thus, assuming this induction hypothesis in cases where $n > 2$, we can conclude, in all cases with $n > 1$, that the bounded infinite sequence $\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3, \ldots$ of points in $\mathbb{R}^{n-1}$ has a convergent subsequence. Let that convergent subsequence be

$$\mathbf{s}_{m_1}, \mathbf{s}_{m_2}, \mathbf{s}_{m_3}, \ldots,$$

where $m_1, m_2, m_3, \ldots$ is a strictly increasing infinite sequence of positive integers, and let $\mathbf{q} = \lim\limits_{j \to +\infty} \mathbf{s}_{m_j}$. There then exists some positive integer L such that

$$|\mathbf{s}_{m_j} - \mathbf{q}| < \tfrac{1}{2}\varepsilon$$

for all positive integers j for which $m_j \geq L$. (Indeed the definition of convergence ensures the existence of a positive integer N that is large enough to ensure that $|\mathbf{s}_{m_j} - \mathbf{q}| < \frac{1}{2}\varepsilon$ whenever $j \geq N$. Taking $L = m_N$ then ensures that $j \geq N$ whenever $m_j \geq L$.)

Let t_j denote the nth component of the point $\mathbf{x}_j$ of $\mathbb{R}^n$ for each positive integer j. The one-dimensional Bolzano-Weierstrass Theorem ensures that the bounded infinite sequence

$$t_{m_1}, t_{m_2}, t_{m_3}, \ldots$$

of real numbers has a convergent subsequence. It follows that there is a strictly increasing infinite sequence $k_1, k_2, k_3, \ldots$ of positive integers, where each k_j is equal to one of the positive integers $m_1, m_2, m_3, \ldots$, such that the infinite sequence

$$t_{k_1}, t_{k_2}, t_{k_3}, \ldots$$

is convergent. Let $r = \lim\limits_{j \to +\infty} t_{k_j}$. There then exists some positive integer M such that $M \geq L$ and

$$|t_{k_j} - r| < \tfrac{1}{2}\varepsilon$$

for all positive integers j for which $k_j \geq M$. It follows that if $k_j \geq M$ then

$$|\mathbf{s}_{k_j} - \mathbf{q}| < \tfrac{1}{2}\varepsilon \quad \text{and} \quad |t_{k_j} - r| < \tfrac{1}{2}\varepsilon.$$

Now there is a point $\mathbf{p}$ of $\mathbb{R}^n$, where $\mathbf{p} = (p_1, p_2, \ldots, p_n)$, determined so that the ith components of the point $\mathbf{p}$ of $\mathbb{R}^n$ is equal to the ith component of the point $\mathbf{q}$ of $\mathbb{R}^{n-1}$ for each integer i between 1 and $n-1$ and also the nth component of the point $\mathbf{p}$ is equal to the real number t.

Also it follows from the definition of the Euclidean norm that

$$|\mathbf{x}_{k_j} - \mathbf{p}|^2 = |\mathbf{s}_{k_j} - \mathbf{q}|^2 + |t_{k_j} - r|^2 < \tfrac{1}{2}\varepsilon^2$$

whenever $k_j \geq M$. But then $|\mathbf{x}_{k_j} - \mathbf{p}| < \varepsilon$ for all positive integers j for which $k_j \geq M$. It follows that $\lim\limits_{j \to +\infty} \mathbf{x}_{k_j} = \mathbf{p}$. We conclude therefore that the bounded infinite sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ does indeed have a convergent subsequence. This completes the proof of the Bolzano-Weierstrass Theorem in dimension n for all positive integers n. ∎

3.7 Cauchy Sequences in Euclidean Spaces

Definition An infinite sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ of points of n-dimensional Euclidean space $\mathbb{R}^n$ is said to be a *Cauchy sequence* if, given any strictly positive real number ε, there exists some positive integer N such that $|\mathbf{x}_j - \mathbf{x}_k| < \varepsilon$ for all positive integers j and k satisfying $j \geq N$ and $k \geq N$.

Lemma 3.6 *Every Cauchy sequence of points of n-dimensional Euclidean space $\mathbb{R}^n$ is bounded.*

Proof Let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ be a Cauchy sequence of points in $\mathbb{R}^n$. Then there exists some positive integer N such that $|\mathbf{x}_j - \mathbf{x}_k| < 1$ whenever $j \geq N$ and $k \geq N$. In particular, $|\mathbf{x}_j| \leq |\mathbf{x}_N| + 1$ whenever $j \geq N$. Therefore $|\mathbf{x}_j| \leq R$ for all positive integers j, where R is the maximum of the real numbers $|\mathbf{x}_1|, |\mathbf{x}_2|, \ldots, |\mathbf{x}_{N-1}|$ and $|\mathbf{x}_N| + 1$. Thus the sequence is bounded, as required. ∎

Theorem 3.7 (Cauchy's Criterion for Convergence) *An infinite sequence of points of n-dimensional Euclidean space $\mathbb{R}^n$ is convergent if and only if it is a Cauchy sequence.*

Proof First we show that convergent sequences in $\mathbb{R}^n$ are Cauchy sequences. Let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ be a convergent sequence of points in $\mathbb{R}^n$, and let $\mathbf{p} = \lim\limits_{j \to +\infty} \mathbf{x}_j$. Let some strictly positive real number ε be given. Then there exists some positive integer N such that $|\mathbf{x}_j - \mathbf{p}| < \frac{1}{2}\varepsilon$ for all $j \geq N$. Thus if $j \geq N$ and $k \geq N$ then $|\mathbf{x}_j - \mathbf{p}| < \frac{1}{2}\varepsilon$ and $|\mathbf{x}_k - \mathbf{p}| < \frac{1}{2}\varepsilon$, and hence

$$|\mathbf{x}_j - \mathbf{x}_k| = |(\mathbf{x}_j - \mathbf{p}) - (\mathbf{x}_k - \mathbf{p})| \leq |\mathbf{x}_j - \mathbf{p}| + |\mathbf{x}_k - \mathbf{p}| < \varepsilon.$$

Thus the sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ is a Cauchy sequence.

Conversely we must show that any Cauchy sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ in $\mathbb{R}^n$ is convergent. Now Cauchy sequences are bounded, by Lemma 3.6. The sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ therefore has a convergent subsequence $\mathbf{x}_{k_1}, \mathbf{x}_{k_2}, \mathbf{x}_{k_3}, \ldots$, by the multidimensional Bolzano-Weierstrass Theorem (Theorem 3.5). Let $\mathbf{p} = \lim_{j \to +\infty} \mathbf{x}_{k_j}$. We claim that the sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ itself converges to $\mathbf{p}$.

Let some strictly positive real number ε be given. Then there exists some positive integer N such that $|\mathbf{x}_j - \mathbf{x}_k| < \frac{1}{2}\varepsilon$ whenever $j \geq N$ and $k \geq N$ (since the sequence is a Cauchy sequence). Let m be chosen large enough to ensure that $k_m \geq N$ and $|\mathbf{x}_{k_m} - \mathbf{p}| < \frac{1}{2}\varepsilon$. Then

$$|\mathbf{x}_j - \mathbf{p}| \leq |\mathbf{x}_j - \mathbf{x}_{k_m}| + |\mathbf{x}_{k_m} - \mathbf{p}| < \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon$$

whenever $j \geq N$. It follows that $\mathbf{x}_j \to \mathbf{p}$ as $j \to +\infty$, as required. ∎