

MAU23203—Analysis in Several Variables
School of Mathematics, Trinity College
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Section 6: The Riemann Integral in One
Dimension

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6. The Riemann Integral in One Dimension

6.1. Darboux Sums and the Riemann Integral

The approach to the theory of integration discussed below was developed by Jean-Gaston Darboux (1842–1917). The integral defined using lower and upper sums in the manner described below is sometimes referred to as the *Darboux integral* of a function on a given interval. However the class of functions that are integrable according to the definitions introduced by Darboux is the class of *Riemann-integrable* functions. Thus the approach using Darboux sums provides a convenient approach to define and establish the basic properties of the *Riemann integral*.

Definition

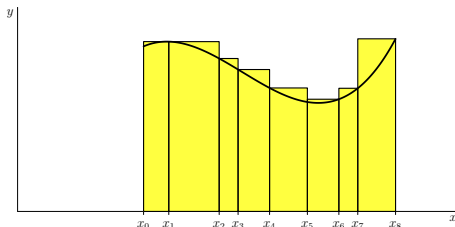
A *partition* P of an interval $[a, b]$ is a set $\{u_0, u_1, u_2, \dots, u_N\}$ of real numbers satisfying $a = u_0 < u_1 < u_2 < \dots < u_{N-1} < u_N = b$.

6. The Riemann Integral in One Dimension (continued)

Given any bounded real-valued function f on $[a, b]$, the *upper sum* (or *upper Darboux sum*) $U(P, f)$ of f for the partition P of $[a, b]$ is defined so that

$$U(P, f) = \sum_{i=1}^N M_i(u_i - u_{i-1}),$$

where $M_i = \sup\{f(x) : u_{i-1} \leq x \leq u_i\}$.

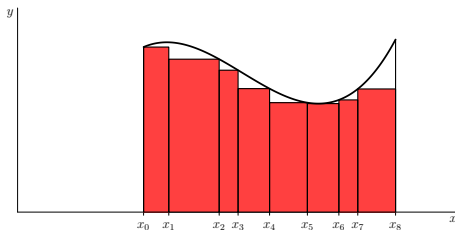


6. The Riemann Integral in One Dimension (continued)

Similarly the *lower sum* (or *lower Darboux sum*) $L(P, f)$ of f for the partition P of $[a, b]$ is defined so that

$$L(P, f) = \sum_{i=1}^N m_i(u_i - u_{i-1}),$$

where $m_i = \inf\{f(x) : u_{i-1} \leq x \leq u_i\}$.



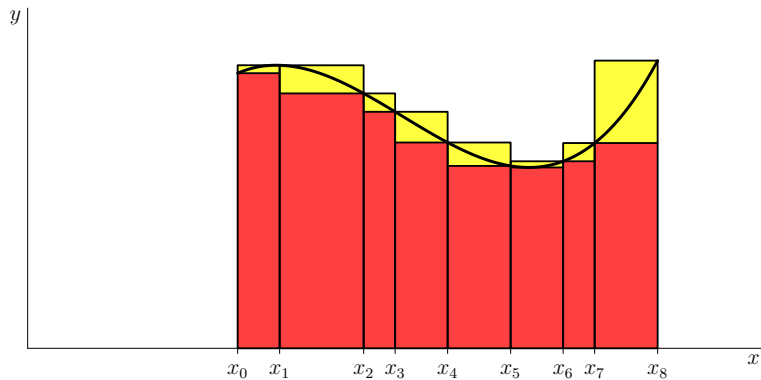
6. The Riemann Integral in One Dimension (continued)

Clearly $L(P, f) \leq U(P, f)$. Moreover $\sum_{i=1}^N (u_i - u_{i-1}) = b - a$, and therefore

$$m(b - a) \leq L(P, f) \leq U(P, f) \leq M(b - a),$$

for any real numbers m and M satisfying $m \leq f(x) \leq M$ for all $x \in [a, b]$.

6. The Riemann Integral in One Dimension (continued)



6. The Riemann Integral in One Dimension (continued)

Definition

Let f be a bounded real-valued function on the interval $[a, b]$, where $a < b$. The *upper Riemann integral* $\mathcal{U} \int_a^b f(x) dx$ (or *upper Darboux integral*) and the *lower Riemann integral* $\mathcal{L} \int_a^b f(x) dx$ (or *lower Darboux integral*) of the function f on $[a, b]$ are defined by

$$\begin{aligned}\mathcal{U} \int_a^b f(x) dx &= \inf \{ U(P, f) : P \text{ is a partition of } [a, b] \}, \\ \mathcal{L} \int_a^b f(x) dx &= \sup \{ L(P, f) : P \text{ is a partition of } [a, b] \}.\end{aligned}$$

The definition of upper and lower integrals thus requires that $\mathcal{U} \int_a^b f(x) dx$ be the infimum of the values of $U(P, f)$ and that $\mathcal{L} \int_a^b f(x) dx$ be the supremum of the values of $L(P, f)$ as P ranges over all possible partitions of the interval $[a, b]$.

Definition

A bounded function $f: [a, b] \rightarrow \mathbb{R}$ on a closed bounded interval $[a, b]$ is said to be *Riemann-integrable* (or *Darboux-integrable*) on $[a, b]$ if

$$\mathcal{U} \int_a^b f(x) dx = \mathcal{L} \int_a^b f(x) dx,$$

in which case the *Riemann integral* $\int_a^b f(x) dx$ (or *Darboux integral*) of f on $[a, b]$ is defined to be the common value of $\mathcal{U} \int_a^b f(x) dx$ and $\mathcal{L} \int_a^b f(x) dx$.

6. The Riemann Integral in One Dimension (continued)

When $a > b$ we define

$$\int_a^b f(x) dx = - \int_b^a f(x) dx$$

for all Riemann-integrable functions f on $[b, a]$. We set

$\int_a^b f(x) dx = 0$ when $b = a$.

If f and g are bounded Riemann-integrable functions on the interval $[a, b]$, and if $f(x) \leq g(x)$ for all $x \in [a, b]$, then

$\int_a^b f(x) dx \leq \int_a^b g(x) dx$, since $L(P, f) \leq L(P, g)$ and $U(P, f) \leq U(P, g)$ for all partitions P of $[a, b]$.

6. The Riemann Integral in One Dimension (continued)

Definition

Let P and R be partitions of $[a, b]$, given by $P = \{u_0, u_1, \dots, u_N\}$ and $R = \{v_0, v_1, \dots, v_L\}$. We say that the partition R is a *refinement* of P if $P \subset R$, so that, for each u_i in P , there is some v_j in R with $u_i = v_j$.

Lemma 6.1

Let R be a refinement of some partition P of $[a, b]$. Then

$$L(R, f) \geq L(P, f) \quad \text{and} \quad U(R, f) \leq U(P, f)$$

for any bounded function $f: [a, b] \rightarrow \mathbb{R}$.

6. The Riemann Integral in One Dimension (continued)

Proof

Let $P = \{u_0, u_1, \dots, u_N\}$ and $R = \{v_0, v_1, \dots, v_L\}$, where $a = u_0 < u_1 < \dots < u_N = b$ and $a = v_0 < v_1 < \dots < v_L = b$.

Now for each integer i between 0 and N there exists some integer $j(i)$ between 0 and L such that $u_i = v_{j(i)}$ for each i , since R is a refinement of P . Moreover

$0 = j(0) < j(1) < \dots < j(N) = L$. For each i , let R_i be the partition of $[u_{i-1}, u_i]$ given by $R_i = \{v_j : j(i-1) \leq j \leq j(i)\}$. Then

$L(R, f) = \sum_{i=1}^N L(R_i, f)$ and $U(R, f) = \sum_{i=1}^N U(R_i, f)$. Moreover

$$m_i(u_i - u_{i-1}) \leq L(R_i, f) \leq U(R_i, f) \leq M_i(u_i - u_{i-1}),$$

since $m_i \leq f(x) \leq M_i$ for all $x \in [u_{i-1}, u_i]$. On summing these inequalities over i , we deduce that

$L(P, f) \leq L(R, f) \leq U(R, f) \leq U(P, f)$, as required. ■

6. The Riemann Integral in One Dimension (continued)

Given any two partitions P and Q of $[a, b]$ there exists a partition R of $[a, b]$ which is a refinement of both P and Q . For example, we can take $R = P \cup Q$. Such a partition is said to be a *common refinement* of the partitions P and Q .

Lemma 6.2

Let f be a bounded real-valued function on the interval $[a, b]$.
Then

$$\mathcal{L} \int_a^b f(x) dx \leq \mathcal{U} \int_a^b f(x) dx.$$

Proof

Let P and Q be partitions of $[a, b]$, and let R be a common refinement of P and Q . It follows from Lemma 6.1 that $L(P, f) \leq L(R, f) \leq U(R, f) \leq U(Q, f)$. Thus, on taking the supremum of the left hand side of the inequality $L(P, f) \leq U(Q, f)$ as P ranges over all possible partitions of the interval $[a, b]$, we see that $\mathcal{L} \int_a^b f(x) dx \leq U(Q, f)$ for all partitions Q of $[a, b]$. But then, taking the infimum of the right hand side of this inequality as Q ranges over all possible partitions of $[a, b]$, we see that $\mathcal{L} \int_a^b f(x) dx \leq \mathcal{U} \int_a^b f(x) dx$, as required. ■

Example

Let $f(x) = cx + d$, where $c \geq 0$. We shall show that f is Riemann-integrable on $[0, 1]$ and evaluate $\int_0^1 f(x) dx$ from first principles.

For each positive integer N , let P_N denote the partition of $[0, 1]$ into N subintervals of equal length. Thus $P_N = \{u_0, u_1, \dots, u_N\}$, where $u_i = i/N$. Now the function f takes values between $(i-1)c/N + d$ and $ic/N + d$ on the interval $[u_{i-1}, u_i]$, and therefore

$$m_i = \frac{(i-1)c}{N} + d, \quad M_i = \frac{ic}{N} + d$$

where $m_i = \inf\{f(x) : u_{i-1} \leq x \leq u_i\}$ and $M_i = \sup\{f(x) : u_{i-1} \leq x \leq u_i\}$. Thus

6. The Riemann Integral in One Dimension (continued)

$$\begin{aligned}L(P_N, f) &= \sum_{i=1}^N m_i(u_i - u_{i-1}) = \frac{1}{N} \sum_{i=1}^N \left(\frac{ci}{N} + d - \frac{c}{N} \right) \\&= \frac{c(N+1)}{2N} + d - \frac{c}{N} = \frac{c}{2} + d - \frac{c}{2N}, \\U(P_N, f) &= \sum_{i=1}^N M_i(u_i - u_{i-1}) = \frac{1}{N} \sum_{i=1}^N \left(\frac{ci}{N} + d \right) \\&= \frac{c(N+1)}{2N} + d = \frac{c}{2} + d + \frac{c}{2N}.\end{aligned}$$

6. The Riemann Integral in One Dimension (continued)

It follows that

$$\lim_{N \rightarrow +\infty} L(P_N, f) = \frac{c}{2} + d$$

and

$$\lim_{N \rightarrow +\infty} U(P_N, f) = \frac{c}{2} + d$$

Now $L(P_N, f) \leq \mathcal{L} \int_a^b f(x) dx \leq \mathcal{U} \int_a^b f(x) dx \leq U(P_N, f)$ for all positive integers N . It follows that

$\mathcal{L} \int_a^b f(x) dx = \frac{1}{2}c + d = \mathcal{U} \int_a^b f(x) dx$. Thus f is Riemann-integrable on the interval $[0, 1]$, and $\int_0^1 f(x) dx = \frac{1}{2}c + d$.

6. The Riemann Integral in One Dimension (continued)

Example

Let $f: [0, 1] \rightarrow \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational;} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

Let P be a partition of the interval $[0, 1]$ given by

$P = \{u_0, u_1, u_2, \dots, u_N\}$, where $0 = u_0 < u_1 < u_2 < \dots < u_N = 1$.

Then

$$\inf\{f(x) : u_{i-1} \leq x \leq u_i\} = 0, \quad \sup\{f(x) : u_{i-1} \leq x \leq u_i\} = 1,$$

for $i = 1, 2, \dots, N$, and thus $L(P, f) = 0$ and $U(P, f) = 1$ for all partitions P of the interval $[0, 1]$. It follows that $\mathcal{L} \int_0^1 f(x) dx = 0$ and $\mathcal{U} \int_0^1 f(x) dx = 1$, and therefore the function f is not Riemann-integrable on the interval $[0, 1]$.

6.2. Basic Properties of the Riemann Integral

Lemma 6.3

Let $f: [a, b] \rightarrow \mathbb{R}$ be a bounded function on a closed bounded interval $[a, b]$, where a and b are real numbers satisfying $a \leq b$. Then the lower and upper Riemann integrals of f and $-f$ are related by the identities

$$\begin{aligned}\mathcal{U} \int_a^b (-f(x)) \, dx &= -\mathcal{L} \int_a^b f(x) \, dx, \\ \mathcal{L} \int_a^b (-f(x)) \, dx &= -\mathcal{U} \int_a^b f(x) \, dx.\end{aligned}$$

6. The Riemann Integral in One Dimension (continued)

Proof

Let P be a partition of $[a, b]$, and let $P = \{u_0, u_1, u_2, \dots, u_N\}$, where

$$a = u_0 < u_1 < u_2 < \dots < u_N = b.$$

Also let

$$m_i[f] = \inf\{f(x) : u_{i-1} \leq x \leq u_i\},$$

$$M_i[f] = \sup\{f(x) : u_{i-1} \leq x \leq u_i\},$$

$$m_i[-f] = \inf\{-f(x) : u_{i-1} \leq x \leq u_i\},$$

$$M_i[-f] = \sup\{-f(x) : u_{i-1} \leq x \leq u_i\}$$

for $i = 1, 2, \dots, N$. Then $m_i[-f] = -M_i[f]$ and $M_i[-f] = -m_i[f]$, and therefore

$$\begin{aligned} L(P, -f) &= \sum_{i=1}^N m_i[-f](u_i - u_{i-1}) = - \sum_{i=1}^N M_i[f](u_i - u_{i-1}) \\ &= -U(P, f). \end{aligned}$$

6. The Riemann Integral in One Dimension (continued)

Thus $L(P, -f) = -U(P, f)$ for all partitions P of the interval $[a, b]$. Similarly $U(P, -f) = -L(P, f)$ for all partitions P of that interval. It follows from the definition of the upper and lower integrals that

$$\begin{aligned}\mathcal{U} \int_a^b (-f(x)) \, dx &= \inf \{ U(P, -f) : P \text{ is a partition of } [a, b] \} \\ &= \inf \{ -L(P, f) : P \text{ is a partition of } [a, b] \} \\ &= -\sup \{ L(P, f) : P \text{ is a partition of } [a, b] \} \\ &= -\mathcal{L} \int_a^b f(x) \, dx\end{aligned}$$

Similarly

$$\begin{aligned}\mathcal{L} \int_a^b (-f(x)) dx &= \sup \{L(P, -f) : P \text{ is a partition of } [a, b]\} \\ &= \sup \{-U(P, f) : P \text{ is a partition of } [a, b]\} \\ &= -\inf \{U(P, f) : P \text{ is a partition of } [a, b]\} \\ &= -\mathcal{U} \int_a^b f(x) dx.\end{aligned}$$

This completes the proof.

Proposition 6.4

Let $f: [a, b] \rightarrow \mathbb{R}$ and $g: [a, b] \rightarrow \mathbb{R}$ be bounded Riemann-integrable functions on a closed bounded interval $[a, b]$, where a and b are real numbers satisfying $a \leq b$. Then the functions $f + g$ and $f - g$ are Riemann-integrable on $[a, b]$, and moreover

$$\int_a^b (f(x) + g(x)) \, dx = \int_a^b f(x) \, dx + \int_a^b g(x) \, dx,$$

and

$$\int_a^b (f(x) - g(x)) \, dx = \int_a^b f(x) \, dx - \int_a^b g(x) \, dx.$$

Proof

Let some strictly positive real number ε be given. The definition of Riemann-integrability and the Riemann integral ensures that there exist partitions P_1 , P_2 , P_3 and P_4 of $[a, b]$ for which

$$L(P_1, f) > \int_a^b f(x) dx - \frac{1}{2}\varepsilon,$$

$$U(P_2, f) < \int_a^b f(x) dx + \frac{1}{2}\varepsilon,$$

$$L(P_3, g) > \int_a^b g(x) dx - \frac{1}{2}\varepsilon$$

and

$$U(P_4, g) < \int_a^b g(x) dx + \frac{1}{2}\varepsilon.$$

6. The Riemann Integral in One Dimension (continued)

Let P be a common refinement of P_1 , P_2 , P_3 and P_4 . Applying Lemma 6.1, we see that

$$L(P, f) \geq L(P_1, f) > \int_a^b f(x) dx - \frac{1}{2}\varepsilon,$$

$$U(P, f) \leq U(P_2, f) < \int_a^b f(x) dx + \frac{1}{2}\varepsilon,$$

$$L(P, g) \geq L(P_3, g) > \int_a^b g(x) dx - \frac{1}{2}\varepsilon$$

and

$$U(P, g) \leq U(P_4, g) < \int_a^b g(x) dx + \frac{1}{2}\varepsilon.$$

6. The Riemann Integral in One Dimension (continued)

Let $P = \{u_0, u_1, \dots, u_N\}$, where

$$a = u_0 < u_1 < \dots < u_N = b,$$

and let

$$M_i[f] = \sup\{f(x) : u_{i-1} \leq x \leq u_i\},$$

$$m_i[f] = \inf\{f(x) : u_{i-1} \leq x \leq u_i\},$$

$$M_i[g] = \sup\{g(x) : u_{i-1} \leq x \leq u_i\},$$

$$m_i[g] = \inf\{g(x) : u_{i-1} \leq x \leq u_i\},$$

$$M_i[f + g] = \sup\{f(x) + g(x) : u_{i-1} \leq x \leq u_i\},$$

$$m_i[f + g] = \inf\{f(x) + g(x) : u_{i-1} \leq x \leq u_i\}.$$

6. The Riemann Integral in One Dimension (continued)

Now the inequalities

$$m_i[f] + m_i[g] \leq f(x) + g(x) \leq M_i[f] + M_i[g]$$

are satisfied for $i = 1, 2, \dots, N$ and for all $x \in [u_{i-1}, u_i]$. It follows from the definitions of $M_i[f + g]$ and $m_i[f + g]$ as the least upper bound and greatest lower bound respectively of the values of $f(x) + g(x)$ on the interval $[u_{i-1}, u_i]$ that

$$m_i[f] + m_i[g] \leq m_i[f + g] \leq M_i[f + g] \leq M_i[f] + M_i[g]$$

for $i = 1, 2, \dots, N$. Multiplying these inequalities by the lengths $u_i - u_{i-1}$ of the subintervals determined by the partition P , and then summing over $i = 1, 2, \dots, N$, we deduce that

$$L(P, f) + L(P, g) \leq L(P, f + g) \leq U(P, f + g) \leq U(P, f) + U(P, g).$$

6. The Riemann Integral in One Dimension (continued)

Now inequalities satisfied by the Darboux upper and lower sums for the partition P guaranteed by the choice of P (as described above) then ensure that

$$L(P, f) + L(P, g) > \int_a^b f(x) dx + \int_a^b g(x) dx - \varepsilon$$

and

$$U(P, f) + U(P, g) < \int_a^b f(x) dx + \int_a^b g(x) dx + \varepsilon.$$

It follows that

$$\begin{aligned} \int_a^b f(x) dx + \int_a^b g(x) dx - \varepsilon &< L(P, f + g) \leq U(P, f + g) \\ &< \int_a^b f(x) dx + \int_a^b g(x) dx + \varepsilon. \end{aligned}$$

6. The Riemann Integral in One Dimension (continued)

But

$$\begin{aligned} L(P, f + g) &\leq \mathcal{L} \int_a^b (f(x) + g(x)) dx \\ &\leq \mathcal{U} \int_a^b (f(x) + g(x)) dx \leq U(P, f + g). \end{aligned}$$

It follows therefore that

$$\begin{aligned} \int_a^b f(x) dx + \int_a^b g(x) dx - \varepsilon &< \mathcal{L} \int_a^b (f(x) + g(x)) dx \\ &\leq \mathcal{U} \int_a^b (f(x) + g(x)) dx \\ &< \int_a^b f(x) dx + \int_a^b g(x) dx + \varepsilon. \end{aligned}$$

6. The Riemann Integral in One Dimension (continued)

These latter inequalities must hold for all positive real numbers ε , no matter how small. It follows that

$$\begin{aligned}\int_a^b f(x) dx + \int_a^b g(x) dx &\leq \mathcal{L} \int_a^b (f(x) + g(x)) dx \\ &\leq \mathcal{U} \int_a^b (f(x) + g(x)) dx \\ &\leq \int_a^b f(x) dx + \int_a^b g(x) dx.\end{aligned}$$

The extreme left hand and extreme right hand sides of the above chain of inequalities are equal. Therefore

$$\begin{aligned}\mathcal{L} \int_a^b (f(x) + g(x)) dx &= \mathcal{U} \int_a^b (f(x) + g(x)) dx \\ &= \int_a^b f(x) dx + \int_a^b g(x) dx.\end{aligned}$$

6. The Riemann Integral in One Dimension (continued)

We conclude therefore that the function $f + g$ is Riemann-integrable and that the value of the Riemann integral of this function is the sum of the integrals of the functions f and g on the interval $[a, b]$.

On replacing g by $-g$, we may deduce the corresponding result for the function $f - g$, thereby completing the proof. ■

Proposition 6.5

Let $f: [a, b] \rightarrow \mathbb{R}$ be a bounded function on a closed bounded interval $[a, b]$, where a and b are real numbers satisfying $a \leq b$. Then the function f is Riemann-integrable on $[a, b]$ if and only if, given any positive real number ε , there exists a partition P of $[a, b]$ with the property that

$$U(P, f) - L(P, f) < \varepsilon.$$

6. The Riemann Integral in One Dimension (continued)

Proof

First suppose that $f: [a, b] \rightarrow \mathbb{R}$ is Riemann-integrable on $[a, b]$. Let some positive real number ε be given. Then

$$\int_a^b f(x) dx$$

is equal to the common value of the lower and upper integrals of the function f on $[a, b]$, and therefore there exist partitions Q and R of $[a, b]$ for which

$$L(Q, f) > \int_a^b f(x) dx - \frac{1}{2}\varepsilon$$

and

$$U(R, f) < \int_a^b f(x) dx + \frac{1}{2}\varepsilon.$$

Let P be a common refinement of the partitions Q and R . Now

$$L(Q, f) \leq L(P, f) \leq U(P, f) \leq U(R, f).$$

(see Lemma 6.1). It follows that

$$U(P, f) - L(P, f) \leq U(R, f) - L(Q, f) < \varepsilon.$$

6. The Riemann Integral in One Dimension (continued)

Now suppose that $f: [a, b] \rightarrow \mathbb{R}$ is a bounded function on $[a, b]$ with the property that, given any positive real number ε , there exists a partition P of $[a, b]$ for which $U(P, f) - L(P, f) < \varepsilon$. Let $\varepsilon > 0$ be given. Then there exists a partition P of $[a, b]$ for which $U(P, f) - L(P, f) < \varepsilon$. Now it follows from the definitions of the upper and lower integrals that

$$L(P, f) \leq \mathcal{L} \int_a^b f(x) dx \leq \mathcal{U} \int_a^b f(x) dx \leq U(P, f),$$

and therefore

$$\mathcal{U} \int_a^b f(x) dx - \mathcal{L} \int_a^b f(x) dx < U(P, f) - L(P, f) < \varepsilon.$$

6. The Riemann Integral in One Dimension (continued)

Thus the difference between the values of the upper and lower integrals of f on $[a, b]$ must be less than every strictly positive real number ε , and therefore

$$\mathcal{U} \int_a^b f(x) dx = \mathcal{L} \int_a^b f(x) dx.$$

This completes the proof. ■

Lemma 6.6

Let f_1, f_2, \dots, f_s and h be bounded real-valued functions on a closed bounded interval $[a, b]$, where a and b are real numbers satisfying $a < b$. Suppose that there exists a positive constant K with the property that

$$|h(v) - h(w)| \leq K \sum_{j=1}^s |f_j(v) - f_j(w)|$$

for all $v, w \in [a, b]$. Then the upper and lower Darboux sums of these real-valued functions satisfy the inequality

$$U(P, h) - L(P, h) \leq K \sum_{j=1}^s (U(P, f_j) - L(P, f_j))$$

for all partitions P of the interval $[a, b]$.

6. The Riemann Integral in One Dimension (continued)

Proof

Let P be a partition of $[a, b]$, and let $P = \{u_0, u_1, \dots, u_N\}$, where

$$a = u_0 < u_1 < \dots < u_N = b,$$

and let

$$M_i[f_j] = \sup\{f_j(x) : u_{i-1} \leq x \leq u_i\},$$

$$m_i[f_j] = \inf\{f_j(x) : u_{i-1} \leq x \leq u_i\}$$

for $j = 1, 2, \dots, s$ and $i = 1, 2, \dots, N$, and

$$M_i[h] = \sup\{h(x) : u_{i-1} \leq x \leq u_i\},$$

$$m_i[h] = \inf\{h(x) : u_{i-1} \leq x \leq u_i\}$$

for $i = 1, 2, \dots, N$.

6. The Riemann Integral in One Dimension (continued)

Let i be an integer between 1 and N . The definitions of $M_i[h]$ and $m_i[h]$ ensure that, given any positive real number δ , there exist $v_i, w_i \in [u_{i-1}, u_i]$ such that $h(v_i) > M_i[h] - \delta$ and $h(w_i) < m_i[h] + \delta$. But then

$$\begin{aligned} M_i[h] - m_i[h] - 2\delta &< h(v_i) - h(w_i) \leq K \sum_{j=1}^s |f_j(v_i) - f_j(w_i)| \\ &\leq K \sum_{j=1}^s (M_i[f_j] - m_i[f_j]). \end{aligned}$$

6. The Riemann Integral in One Dimension (continued)

The inequality

$$M_i[h] - m_i[h] - 2\delta < K \sum_{j=1}^s (M_i[f_j] - m_i[f_j])$$

therefore holds for all positive values of the real number δ , no matter how small, and therefore

$$M_i[h] - m_i[h] \leq K \sum_{j=1}^s (M_i[f_j] - m_i[f_j]).$$

Multiplying both sides of this inequality by the length $u_i - u_{i-1}$ of the i th subinterval of $[a, b]$ determined by the partition P , and summing for $i = 1, 2, \dots, N$, we find that

6. The Riemann Integral in One Dimension (continued)

$$\begin{aligned}U(P, h) - L(P, h) &= \sum_{i=1}^N (M_i[h] - m_i[h])(u_i - u_{i-1}) \\&\leq K \sum_{j=1}^s \sum_{i=1}^N (M_i[f_j] - m_i[f_j])(u_i - u_{i-1}) \\&\leq K \sum_{j=1}^s (U(P, f_j) - L(P, f_j)).\end{aligned}$$

We conclude therefore that

$$U(P, h) - L(P, h) \leq K \sum_{j=1}^s (U(P, f_j) - L(P, f_j))$$

for all partitions P of the interval $[a, b]$, as required. ■

Proposition 6.7

Let f_1, f_2, \dots, f_s be bounded Riemann-integrable real-valued functions on a closed bounded interval $[a, b]$, where a and b are real numbers satisfying $a < b$, and let h be a bounded real-valued function on $[a, b]$. Suppose that there exists a positive constant K with the property that

$$|h(v) - h(w)| \leq K \sum_{j=1}^s |f_j(v) - f_j(w)|$$

for all $u, v \in [a, b]$. Then the function h is Riemann-integrable on $[a, b]$.

6. The Riemann Integral in One Dimension (continued)

Proof

Given any positive real number ε , there exist partitions P_1, P_2, \dots, P_s of $[a, b]$ with the property that

$$U(P_j, f_j) - L(P_j, f_j) < \frac{\varepsilon}{sK}$$

for $j = 1, 2, \dots, s$ (see Proposition 6.5). Let P be a common refinement of the partitions P_1, P_2, \dots, P_s . Then

$$U(P, f_j) - L(P, f_j) \leq U(P_j, f_j) - L(P_j, f_j) < \frac{\varepsilon}{sK}$$

for $j = 1, 2, \dots, s$ (see Lemma 6.1). It then follows from Lemma 6.6 that

$$U(P, h) - L(P, h) \leq K \sum_{j=1}^s (U(P, f_j) - L(P, f_j)) < \varepsilon.$$

On applying Proposition 6.5, we therefore conclude that the function h is Riemann-integrable on $[a, b]$, as required. ■

Proposition 6.8

Let $f: [a, b] \rightarrow \mathbb{R}$ and $g: [a, b] \rightarrow \mathbb{R}$ be bounded Riemann-integrable functions on a closed bounded interval $[a, b]$, where a and b are real numbers satisfying $a \leq b$. Then the function $f \cdot g$ is Riemann-integrable on $[a, b]$, where $(f \cdot g)(x) = f(x)g(x)$ for all $x \in [a, b]$.

Proof

The functions f and g are bounded on $[a, b]$, and therefore there exists some positive real number K with the property that $|f(x)| \leq K$ and $|g(x)| \leq K$ for all $x \in [a, b]$. But then

$$\begin{aligned} & |f(v)g(v) - f(w)g(w)| \\ &= |f(v)(g(v) - g(w)) + (f(v) - f(w))g(w)| \\ &\leq |f(v)(g(v) - g(w))| + |(f(v) - f(w))g(w)| \\ &\leq K(|g(v) - g(w)| + |f(v) - f(w)|) \end{aligned}$$

for all $v, w \in [a, b]$. The result therefore follows directly on applying Proposition 6.7. ■

Proposition 6.9

Let $f: [a, b] \rightarrow \mathbb{R}$ be a bounded Riemann-integrable function on a closed interval $[a, b]$, where a and b are real numbers satisfying $a \leq b$, and let $|f|: [a, b] \rightarrow \mathbb{R}$ be the function defined such that $|f|(x) = |f(x)|$ for all $x \in [a, b]$. Then the function $|f|$ is Riemann-integrable on $[a, b]$, and

$$\left| \int_a^b f(x) \, dx \right| \leq \int_a^b |f(x)| \, dx.$$

Proof

Let some positive real number ε be given. It follows from Proposition 6.5 that there exists a partition P of $[a, b]$ such that

$$U(P, f) - L(P, f) < \varepsilon.$$

$$\left| |f(v)| - |f(w)| \right| \leq |f(v) - f(w)|$$

for all $v, w \in [a, b]$. Applying Lemma 6.6, we conclude that

$$U(P, |f|) - L(P, |f|) \leq U(P, f) - L(P, f) < \varepsilon.$$

Proposition 6.5 then ensures that the function $|f|$ is Riemann-integrable on $[a, b]$,

6. The Riemann Integral in One Dimension (continued)

Now $-|f(x)| \leq f(x) \leq |f(x)|$ for all $x \in [a, b]$. It follows that

$$-\int_a^b |f(x)| \, dx \leq \int_a^b f(x) \, dx \leq \int_a^b |f(x)| \, dx.$$

It follows that

$$\left| \int_a^b f(x) \, dx \right| \leq \int_a^b |f(x)| \, dx,$$

as required. ■

Proposition 6.10

Let f be a bounded real-valued function on the interval $[a, c]$. Suppose that f is Riemann-integrable on the intervals $[a, b]$ and $[b, c]$, where $a < b < c$. Then f is Riemann-integrable on $[a, c]$, and

$$\int_a^c f(x) dx = \int_a^b f(x) dx + \int_b^c f(x) dx.$$

Proof

Let some positive real number ε be given. The function f is Riemann-integrable on the interval $[a, b]$ and therefore there exists a partition Q of $[a, b]$ such that the lower Darboux sum $L(Q, f)$ of f on $[a, b]$ with respect to the partition Q of $[a, b]$ satisfies

$$L(Q, f) > \int_a^b f(x) dx - \frac{1}{2}\varepsilon.$$

Similarly there exists a partition R of $[b, c]$ of $[a, b]$ such that the lower Darboux sum $L(R, f)$ of f on $[b, c]$ with respect to the partition R of $[b, c]$ satisfies

$$L(R, f) > \int_b^c f(x) dx - \frac{1}{2}\varepsilon.$$

6. The Riemann Integral in One Dimension (continued)

Now the partitions Q and R combine to give a partition P of the interval $[a, c]$, where $P = Q \cup R$. Indeed $Q = \{v_0, v_1, \dots, v_L\}$, where v_0, v_1, \dots, v_L are real numbers satisfying

$$a = v_0 < v_1 < v_2 < \cdots < v_{L-1} < v_L = b,$$

and $R = \{w_0, w_1, \dots, w_N\}$, where w_0, w_1, \dots, w_N are real numbers satisfying

$$b = w_0 < w_1 < w_2 < \cdots < w_{N-1} < w_N = c.$$

Then

$$P = \{a, v_1, v_2, \dots, v_{L-1}, b, w_1, w_2, \dots, w_{N-1}, c\}.$$

It follows directly from the definition of Darboux lower sums that

$$L(P, f) = L(Q, f) + L(R, f).$$

6. The Riemann Integral in One Dimension (continued)

The choice of the partitions Q and R then ensures that

$$L(P, f) > \int_a^b f(x) dx + \int_b^c f(x) dx - \varepsilon.$$

The lower Riemann integral $\mathcal{L} \int_a^c f(x) dx$ is by definition the least upper bound of the lower Darboux sums of f on the interval $[a, c]$. It follows that

$$\mathcal{L} \int_a^c f(x) dx > \int_a^b f(x) dx + \int_b^c f(x) dx - \varepsilon.$$

Moreover this inequality holds for all values of the positive real number ε . It follows that

$$\mathcal{L} \int_a^c f(x) dx \geq \int_a^b f(x) dx + \int_b^c f(x) dx.$$

6. The Riemann Integral in One Dimension (continued)

Applying this result with the function f replaced by $-f$ yields the inequality

$$\mathcal{L} \int_a^c (-f(x)) \, dx \geq - \int_a^b f(x) \, dx - \int_b^c f(x) \, dx.$$

But

$$\mathcal{L} \int_a^c (-f(x)) \, dx = -\mathcal{U} \int_a^c f(x) \, dx$$

(see Lemma 6.3). It follows that

$$\mathcal{U} \int_a^c f(x) \, dx \leq \int_a^b f(x) \, dx + \int_b^c f(x) \, dx \leq \mathcal{L} \int_a^c f(x) \, dx.$$

6. The Riemann Integral in One Dimension (continued)

But

$$\mathcal{L} \int_a^c f(x) dx \leq \mathcal{U} \int_a^c f(x) dx.$$

It follows that

$$\mathcal{L} \int_a^c f(x) dx = \mathcal{U} \int_a^c f(x) dx = \int_a^b f(x) dx + \int_b^c f(x) dx.$$

The result follows. ■

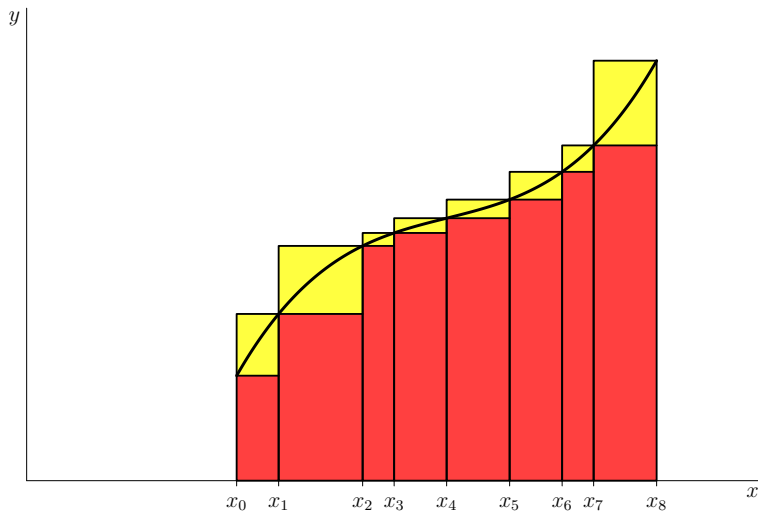
6.3. Integrability of Monotonic Functions

Let a and b be real numbers satisfying $a < b$. A real-valued function $f: [a, b] \rightarrow \mathbb{R}$ defined on the closed bounded interval $[a, b]$ is said to be *non-decreasing* if $f(v) \leq f(w)$ for all real numbers v and w satisfying $a \leq v \leq w \leq b$. Similarly $f: [a, b] \rightarrow \mathbb{R}$ is said to be *non-increasing* if $f(v) \geq f(w)$ for all real numbers v and w satisfying $a \leq v \leq w \leq b$. The function $f: [a, b] \rightarrow \mathbb{R}$ is said to be *monotonic* on $[a, b]$ if either it is non-decreasing on $[a, b]$ or else it is non-increasing on $[a, b]$.

Proposition 6.11

Let a and b be real numbers satisfying $a < b$. Then every monotonic function on the interval $[a, b]$ is Riemann-integrable on $[a, b]$.

6. The Riemann Integral in One Dimension (continued)



Proof

Let $f: [a, b] \rightarrow \mathbb{R}$ be a non-decreasing function on the closed bounded interval $[a, b]$. Then $f(a) \leq f(x) \leq f(b)$ for all $x \in [a, b]$, and therefore the function f is bounded on $[a, b]$. Let some positive real number ε be given. Let δ be some strictly positive real number for which $(f(b) - f(a))\delta < \varepsilon$, and let P be a partition of $[a, b]$ of the form $P = \{u_0, u_1, u_2, \dots, u_N\}$, where

$$a = u_0 < u_1 < u_2 < \dots < u_{N-1} < u_N = b$$

and $u_i - u_{i-1} < \delta$ for $i = 1, 2, \dots, N$.

6. The Riemann Integral in One Dimension (continued)

The maximum and minimum values of $f(x)$ on the interval $[u_{i-1}, u_i]$ are attained at u_i and u_{i-1} respectively, and therefore the upper sum $U(P, f)$ and $L(P, f)$ of f for the partition P satisfy

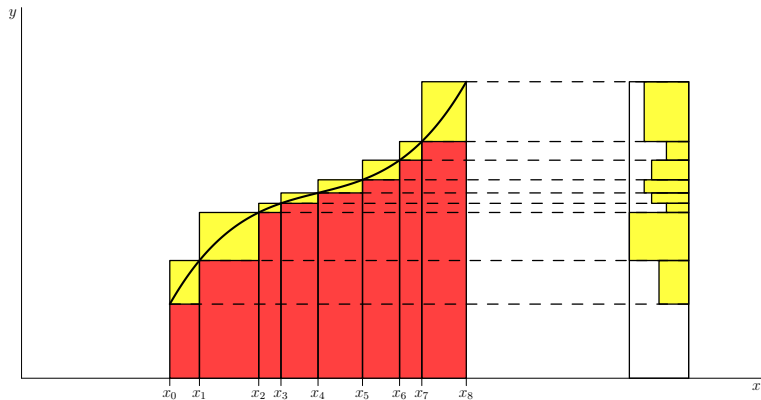
$$U(P, f) = \sum_{i=1}^N f(u_i)(u_i - u_{i-1})$$

and

$$L(P, f) = \sum_{i=1}^N f(u_{i-1})(u_i - u_{i-1}).$$

Now $f(u_i) - f(u_{i-1}) \geq 0$ for $i = 1, 2, \dots, N$. It follows that

6. The Riemann Integral in One Dimension (continued)



6. The Riemann Integral in One Dimension (continued)

$$\begin{aligned} & U(P, f) - L(P, f) \\ &= \sum_{i=1}^N (f(u_i) - f(u_{i-1}))(u_i - u_{i-1}) \\ &< \delta \sum_{i=1}^N (f(u_i) - f(u_{i-1})) = \delta(f(b) - f(a)) < \varepsilon. \end{aligned}$$

We have thus shown that

$$\mathcal{U} \int_a^b f(x) dx - \mathcal{L} \int_a^b f(x) dx < \varepsilon$$

for all strictly positive numbers ε . But

$$\mathcal{U} \int_a^b f(x) dx \geq \mathcal{L} \int_a^b f(x) dx.$$

It follows that

$$\mathcal{U} \int_a^b f(x) dx = \mathcal{L} \int_a^b f(x) dx,$$

and thus the function f is Riemann-integrable on $[a, b]$.

Now let $f: [a, b] \rightarrow \mathbb{R}$ be a non-increasing function on $[a, b]$. Then $-f$ is a non-decreasing function on $[a, b]$ and it follows from what we have just shown that $-f$ is Riemann-integrable on $[a, b]$. It follows that the function f itself must be Riemann-integrable on $[a, b]$, as required. ■

Corollary 6.12

Let $f: [a, b] \rightarrow \mathbb{R}$ be a real-valued function on the interval $[a, b]$, where a and b are real numbers satisfying $a < b$. Suppose that there exist real numbers u_0, u_1, \dots, u_N , where

$$a = u_0 < u_1 < u_2 < \dots < u_{N-1} < u_N = b,$$

such that the function f restricted to the interval $[u_{i-1}, u_i]$ is monotonic on $[u_{i-1}, u_i]$ for $i = 1, 2, \dots, N$. Then f is Riemann-integrable on $[a, b]$.

Proof

The result follows immediately on applying the results of Proposition 6.10 and Proposition 6.11. ■

Remark

The result and proof-strategy of Proposition 6.11 are to be found in their essentials in Isaac Newton, *Philosophiae naturalis principia mathematica* (1686), Book 1, Section 1, Lemmas 2 and 3.

6.4. Integrability of Continuous functions

Theorem 6.13

Let a and b be real numbers satisfying $a < b$. Then any continuous real-valued function on the interval $[a, b]$ is Riemann-integrable.

Proof

Let f be a continuous real-valued function on $[a, b]$. Then f is bounded above and below on the interval $[a, b]$, and moreover $f: [a, b] \rightarrow \mathbb{R}$ is uniformly continuous on $[a, b]$. (These results follow from Theorem 4.21 and Theorem 4.22.) Therefore there exists some strictly positive real number δ such that $|f(x) - f(y)| < \varepsilon$ whenever $x, y \in [a, b]$ satisfy $|x - y| < \delta$.

6. The Riemann Integral in One Dimension (continued)

Choose a partition P of the interval $[a, b]$ such that each subinterval in the partition has length less than δ . Write $P = \{u_0, u_1, \dots, u_N\}$, where $a = u_0 < u_1 < \dots < u_N = b$. Now if $u_{i-1} \leq x \leq u_i$ then $|x - u_i| < \delta$, and hence $f(u_i) - \varepsilon < f(x) < f(u_i) + \varepsilon$. It follows that

$$f(u_i) - \varepsilon \leq m_i \leq M_i \leq f(u_i) + \varepsilon \quad (i = 1, 2, \dots, N),$$

where $m_i = \inf\{f(x) : u_{i-1} \leq x \leq u_i\}$ and $M_i = \sup\{f(x) : u_{i-1} \leq x \leq u_i\}$. Therefore

$$\begin{aligned} \sum_{i=1}^N f(u_i)(u_i - u_{i-1}) - \varepsilon(b - a) \\ \leq L(P, f) \leq U(P, f) \\ \leq \sum_{i=1}^N f(u_i)(u_i - u_{i-1}) + \varepsilon(b - a), \end{aligned}$$

where $L(P, f)$ and $U(P, f)$ denote the lower and upper sums of the function f for the partition P .

6. The Riemann Integral in One Dimension (continued)

We have now shown that

$$0 \leq \mathcal{U} \int_a^b f(x) dx - \mathcal{L} \int_a^b f(x) dx \leq U(P, f) - L(P, f) \leq 2\varepsilon(b-a).$$

But this inequality must be satisfied for any strictly positive real number ε . Therefore

$$\mathcal{U} \int_a^b f(x) dx = \mathcal{L} \int_a^b f(x) dx,$$

and thus the function f is Riemann-integrable on $[a, b]$. ■

6.5. The Fundamental Theorem of Calculus

Let a and b be real numbers satisfying $a < b$. One can show that all continuous functions on the interval $[a, b]$ are Riemann-integrable (see Theorem 6.13). However the task of calculating the Riemann integral of a continuous function directly from the definition is difficult if not impossible for all but the simplest functions. Thus to calculate such integrals one makes use of the Fundamental Theorem of Calculus.

Theorem 6.14 (The Fundamental Theorem of Calculus)

Let f be a continuous real-valued function on the interval $[a, b]$, where $a < b$. Then

$$\frac{d}{dx} \left(\int_a^x f(t) dt \right) = f(x)$$

for all x satisfying $a < x < b$.

Proof

Let some strictly positive real number ε be given, and let ε_0 be a real number chosen so that $0 < \varepsilon_0 < \varepsilon$. (For example, one could choose $\varepsilon_0 = \frac{1}{2}\varepsilon$.) Now the function f is continuous at x , where $a < x < b$. It follows that there exists some strictly positive real number δ such that

$$f(x) - \varepsilon_0 \leq f(t) \leq f(x) + \varepsilon_0$$

for all $t \in [a, b]$ satisfying $x - \delta < t < x + \delta$.

6. The Riemann Integral in One Dimension (continued)

Let $F(s) = \int_a^s f(t) dt$ for all $s \in (a, b)$. Then

$$\begin{aligned} F(x+h) &= \int_a^{x+h} f(t) dt = \int_a^x f(t) dt + \int_x^{x+h} f(t) dt \\ &= F(x) + \int_x^{x+h} f(t) dt \end{aligned}$$

whenever $x+h \in [a, b]$. Also

$$\frac{1}{h} \int_x^{x+h} f(x) dt = \frac{f(x)}{h} \int_x^{x+h} dt = f(x),$$

because $f(x)$ is constant as t varies between x and $x+h$. It follows that

$$\frac{F(x+h) - F(x)}{h} - f(x) = \frac{1}{h} \int_x^{x+h} (f(t) - f(x)) dt$$

whenever $x+h \in [a, b]$.

6. The Riemann Integral in One Dimension (continued)

But if $0 < |h| < \delta$ and $x + h \in [a, b]$ then

$$-\varepsilon_0 \leq f(t) - f(x) \leq \varepsilon_0$$

for all real numbers t belonging to the closed interval with endpoints x and $x + h$, and therefore

$$-\varepsilon_0 |h| \leq \int_x^{x+h} (f(t) - f(x)) dt \leq \varepsilon_0 |h|.$$

It follows that

$$\left| \frac{F(x+h) - F(x)}{h} - f(x) \right| \leq \varepsilon_0 < \varepsilon$$

whenever $x + h \in [a, b]$ and $0 < |h| < \delta$. We conclude that

$$\frac{d}{dx} \left(\int_a^x f(t) dt \right) = \lim_{h \rightarrow 0} \frac{F(x+h) - F(x)}{h} = f(x),$$

as required. ■