MAU23203—Analysis in Several Variables
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Section 1: The Real Number System

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1.1. A Concise Characterization of the Real Number System

The set $\mathbb{R}$ of *real numbers*, with its usual ordering and algebraic operations of addition and multiplication, is a Dedekind-complete ordered field.

We describe below what a *field* is, what an *ordered field* is, and what is meant by saying that an ordered field is *Dedekind-complete*.

1.2. Fields

Definition

A *field* is a set $\mathbb{F}$ on which are defined operations of addition and multiplication, associating elements $x + y$ and xy of $\mathbb{F}$ to each pair x, y of elements of $\mathbb{F}$, for which the following axioms are satisfied:

(i) $x + y = y + x$ for all $x, y \in \mathbb{F}$ (i.e., the operation of addition on $\mathbb{F}$ is *commutative*);

(ii) $(x + y) + z = x + (y + z)$ for all $x, y, z \in \mathbb{F}$ (i.e., the operation of addition on $\mathbb{F}$ is *associative*);

(iii) there exists an element 0 of $\mathbb{F}$ with the property that $0 + x = x$ for all $x \in \mathbb{F}$ (i.e., there exists a *zero element* for the operation of addition on $\mathbb{F}$);

(iv) given any $x \in \mathbb{F}$, there exists an element $-x$ of $\mathbb{F}$ satisfying $x + (-x) = 0$ (i.e., *negatives* of elements of $\mathbb{F}$ always exist);

(v) $xy = yx$ for all $x, y \in \mathbb{F}$ (i.e., the operation of multiplication on $\mathbb{F}$ is *commutative*);

(vi) $(xy)z = x(yz)$ for all $x, y, z \in \mathbb{F}$ (i.e., the operation of multiplication on $\mathbb{F}$ is *associative*);

(vii) there exists an element 1 of $\mathbb{F}$ with the property that $1x = x$ for all $x \in \mathbb{F}$ (i.e., there exists an *identity element* for the operation of multiplication on $\mathbb{F}$);

(viii) given any $x \in \mathbb{F}$ satisfying $x \neq 0$, there exists an element x^{-1} of $\mathbb{F}$ satisfying $xx^{-1} = 1$;

(ix) $x(y + z) = xy + xz$ for all $x, y, z \in \mathbb{F}$ (i.e., multiplication is *distributive* over addition).

The operations of subtraction and division are defined on a field $\mathbb{F}$ in terms of the operations of addition and multiplication on that field in the obvious fashion: $x - y = x + (-y)$ for all elements x and y of $\mathbb{F}$, and moreover $x/y = xy^{-1}$ provided that $y \neq 0$.

1.3. Ordered Fields

Definition

An *ordered field* consists of a field $\mathbb{F}$ together with an ordering $<$ on that field that satisfies the following axioms:—

(x) if x and y are elements of $\mathbb{F}$ then one and only one of the three statements $x < y$, $x = y$ and $y < x$ is true (i.e., the ordering satisfies the *Trichotomy Law*);

(xi) if x, y and z are elements of $\mathbb{F}$ and if $x < y$ and $y < z$ then $x < z$ (i.e., the ordering is *transitive*);

(xii) if x, y and z are elements of $\mathbb{F}$ and if $x < y$ then $x + z < y + z$;

(xiii) if x and y are elements of $\mathbb{F}$ which satisfy $0 < x$ and $0 < y$ then $0 < xy$.

We can write $x > y$ in cases where $y < x$. we can write $x \leq y$ in cases where either $x = y$ or $x < y$. We can write $x \geq y$ in cases where either $x = y$ or $y < x$.

Example

The rational numbers, with the standard ordering, and the standard operations of addition, subtraction, multiplication, and division constitute an ordered field.

Example

Let $\mathbb{Q}(\sqrt{2})$ denote the set of all numbers that can be represented in the form $b + c\sqrt{2}$, where b and c are rational numbers. The sum and difference of any two numbers belonging to $\mathbb{Q}(\sqrt{2})$ themselves belong to $\mathbb{Q}(\sqrt{2})$. Also the product of any two numbers $\mathbb{Q}(\sqrt{2})$ itself belongs to $\mathbb{Q}(\sqrt{2})$ because, for any rational numbers b, c, e and f,

$$(b + c\sqrt{2})(e + f\sqrt{2}) = (be + 2cf) + (bf + ce)\sqrt{2},$$

and both $be + 2cf$ and $bf + ce$ are rational numbers. The reciprocal of any non-zero element of $\mathbb{Q}(\sqrt{2})$ itself belongs to $\mathbb{Q}(\sqrt{2})$, because

$$\frac{1}{b + c\sqrt{2}} = \frac{b - c\sqrt{2}}{b^2 - 2c^2}.$$

for all rational numbers b and c. It is then a straightforward exercise to verify that $\mathbb{Q}(\sqrt{2})$ is an ordered field.

1.4. Least Upper Bounds

Let S be a subset of an ordered field $\mathbb{F}$. An element u of $\mathbb{F}$ is said to be an *upper bound* of the set S if $x \leq u$ for all $x \in S$. The set S is said to be *bounded above* if such an upper bound exists.

Definition

Let $\mathbb{F}$ be an ordered field, and let S be some subset of $\mathbb{F}$ which is bounded above. An element s of $\mathbb{F}$ is said to be the *least upper bound* (or *supremum*) of S (denoted by $\sup S$) if s is an upper bound of S and $s \leq u$ for all upper bounds u of S.

Example

The rational number 2 is the least upper bound, in the ordered field of rational numbers, of the sets $\{x \in \mathbb{Q} : x \leq 2\}$ and $\{x \in \mathbb{Q} : x < 2\}$. Note that the first of these sets contains its least upper bound, whereas the second set does not.

The following property is satisfied in some ordered fields but not in others.

> **Least Upper Bound Property**: *given any non-empty subset S of $\mathbb{F}$ that is bounded above, there exists an element* $\sup S$ *of $\mathbb{F}$ that is the least upper bound for the set S.*

Definition

A *Dedekind-complete* ordered field $\mathbb{F}$ is an ordered field which has the Least Upper Bound Property.

1.5. Greatest Lower Bounds

Let S be a subset of an ordered field $\mathbb{F}$. A *lower bound* of S is an element l of $\mathbb{F}$ with the property that $l \leq x$ for all $x \in S$. The set S is said to be *bounded below* if such a lower bound exists. A *greatest lower bound* (or *infimum*) for a set S is a lower bound for that set that is greater than every other lower bound for that set. The greatest lower bound of the set S (if it exists) is denoted by $\inf S$.

Let $\mathbb{F}$ be a Dedekind-complete ordered field. Then, given any non-empty subset S of $\mathbb{F}$ that is bounded below, there exists a greatest lower bound (or *infimum*) $\inf S$ for the set S. Indeed $\inf S = -\sup\{x \in \mathbb{R} : -x \in S\}$.

Remark

It can be proved that any two Dedekind-complete ordered fields are isomorphic via an isomorphism that respects the ordering and the algebraic operations on the fields. The theory of *Dedekind cuts* provides a construction that yields a Dedekind-complete ordered field that can represent the system of real numbers. For an account of this construction, and for a proof that these axioms are sufficient to characterize the real number system, see chapters 27–29 of *Calculus*, by M. Spivak. The construction of the real number system using Dedekind cuts is also described in detail in the Appendix to Chapter 1 of *Principles of Real Analysis* by W. Rudin.

1.6. Bounded Sets of Real Numbers

The set $\mathbb{R}$ of *real numbers*, with its usual ordering algebraic operations, constitutes a Dedekind-complete ordered field. Thus every non-empty subset S of $\mathbb{R}$ that is bounded above has a *least upper bound* (or *supremum*) $\sup S$, and every non-empty subset S of $\mathbb{R}$ that is bounded below has a *greatest lower bound* (or *infimum*) $\inf S$.

Let S be a non-empty subset of the real numbers that is bounded (both above and below). Then the closed interval $[\inf S, \sup S]$ is the smallest closed interval in the set $\mathbb{R}$ of real numbers that contains the set S. Indeed if $S \subset [a, b]$, where a and b are real numbers satisfying $a \leq b$, then $a \leq \inf S \leq \sup S \leq b$, and therefore

$$S \subset [\inf S, \sup S] \subset [a, b].$$

1.7. Absolute Values of Real Numbers

Let x be a real number. The *absolute value* $|x|$ of x is defined so that

$$|x| = \begin{cases} x & \text{if } x \geq 0; \\ -x & \text{if } x < 0; \end{cases}$$

Lemma 1.1

Let u and v be real numbers. Then $|u + v| \leq |u| + |v|$ and $|uv| = |u|\,|v|$.

Proof

Let u and v be real numbers. Then

$$-|u| \leq u \leq |u| \quad \text{and} \quad -|v| \leq v \leq |v|.$$

On adding inequalities, we find that

$$-(|u| + |v|) = -|u| - |v| \leq u + v \leq |u| + |v|,$$

and thus

$$u + v \leq |u| + |v| \quad \text{and} \quad -(u + v) \leq |u| + |v|.$$

Now the value of $|u+v|$ is equal to at least one of the numbers $u+v$ and $-(u+v)$. It follows that

$$|u+v| \leq |u| + |v|$$

for all real numbers u and v.

Next we note that $|u|\,|v|$ is the product of one or other of the numbers u and $-u$ with one or other of the numbers v and $-v$, and therefore its value is equal either to uv or to $-uv$. Because both $|u|\,|v|$ and $|uv|$ are non-negative, we conclude that $|uv| = |u|\,|v|$, as required. ∎

Lemma 1.2

Let u and v be real numbers. Then $\bigl||u| - |v|\bigr| \leq |u - v|$.

Proof

It follows from Lemma 1.1 that

$$|u| = |v + (u - v)| \leq |v| + |u - v|.$$

Therefore $|u| - |v| \leq |u - v|$. Interchanging u and v, we find also that

$$|v| - |u| \leq |v - u| = |u - v|.$$

Now $\bigl||u| - |v|\bigr|$ is equal to one or other of the real numbers $|u| - |v|$ and $|v| - |u|$. It follows that $\bigl||u| - |v|\bigr| \leq |u - v|$, as required. ∎

1.8. Convergence of Infinite Sequences of Real Numbers

An *infinite sequence* $x_1, x_2, x_3, \ldots$ of real numbers associates to each positive integer j a corresponding real number x_j.

Definition

An infinite sequence $x_1, x_2, x_3, \ldots$ of real numbers is said to *converge* to some real number p if and only if the following criterion is satisfied:

> *given any strictly positive real number ε, there exists some positive integer N such that $|x_j - p| < \varepsilon$ for all positive integers j satisfying $j \geq N$.*

If an infinite sequence $x_1, x_2, x_3, \ldots$ of real numbers converges to some real number p, then p is said to be the *limit* of the sequence, and we can indicate the convergence of the infinite sequence to p by writing '$x_j \to p$ as $j \to +\infty$', or by writing '$\lim_{j \to +\infty} x_j = p$'.

Let x and p be real numbers, and let ε be a strictly positive real number. Then $|x-p| < \varepsilon$ if and only if both $x-p<\varepsilon$ and $p-x<\varepsilon$. It follows that $|x-p|<\varepsilon$ if and only if $p-\varepsilon < x < p+\varepsilon$. The condition $|x-p|<\varepsilon$ essentially requires that the value of the real number x should agree with p to within an error of at most ε. An infinite sequence $x_1, x_2, x_3, \ldots$ of real numbers converges to some real number p if and only if, given any positive real number ε, there exists some positive integer N such that $p-\varepsilon < x_j < p+\varepsilon$ for all positive integers j satisfying $j \geq N$.

Definition

We say that an infinite sequence $x_1, x_2, x_3, \ldots$ of real numbers is *bounded above* if there exists some real number B such that $x_j \leq B$ for all positive integers j. Similarly we say that this sequence is *bounded below* if there exists some real number A such that $x_j \geq A$ for all positive integers j. A sequence is said to be *bounded* if it is bounded above and bounded below. Thus a sequence is bounded if and only if there exist real numbers A and B such that $A \leq x_j \leq B$ for all positive integers j.

Lemma 1.3

Every convergent sequence of real numbers is bounded.

Proof

Let $x_1, x_2, x_3, \ldots$ be a sequence of real numbers converging to some real number p. On applying the formal definition of convergence (with $\varepsilon = 1$), we deduce the existence of some positive integer N such that $p - 1 < x_j < p + 1$ for all $j \geq N$. But then $A \leq x_j \leq B$ for all positive integers j, where A is the minimum of $x_1, x_2, \ldots, x_{N-1}$ and $p - 1$, and B is the maximum of $x_1, x_2, \ldots, x_{N-1}$ and $p + 1$. ∎

Proposition 1.4

Let $x_1, x_2, x_3, \ldots$ and $y_1, y_2, y_3,$ be convergent infinite sequences of real numbers. Then the sum and difference of these sequences are convergent, and

$$\begin{aligned} \lim_{j \to +\infty} (x_j + y_j) &= \lim_{j \to +\infty} x_j + \lim_{j \to +\infty} y_j, \\ \lim_{j \to +\infty} (x_j - y_j) &= \lim_{j \to +\infty} x_j - \lim_{j \to +\infty} y_j. \end{aligned}$$

Proof

Throughout this proof let $p = \lim_{j \to +\infty} x_j$ and $q = \lim_{j \to +\infty} y_j$. It follows directly from the definition of limits that $\lim_{j \to +\infty} (-y_j) = -q$.

Let some strictly positive real number ε be given. We must show that there exists some positive integer N such that $|x_j + y_j - (p+q)| < \varepsilon$ whenever $j \geq N$. Now $x_j \to p$ as $j \to +\infty$, and therefore, given any strictly positive real number ε_1, there exists some positive integer N_1 with the property that $|x_j - p| < \varepsilon_1$ whenever $j \geq N_1$. In particular, there exists a positive integer N_1 with the property that $|x_j - p| < \frac{1}{2}\varepsilon$ whenever $j \geq N_1$. (To see this, let $\varepsilon_1 = \frac{1}{2}\varepsilon$.) Similarly there exists some positive integer N_2 such that $|y_j - q| < \frac{1}{2}\varepsilon$ whenever $j \geq N_2$. Let N be the maximum of N_1 and N_2. If $j \geq N$ then

$$\begin{aligned} |x_j + y_j - (p+q)| &= |(x_j - p) + (y_j - q)| \leq |x_j - p| + |y_j - q| \\ &< \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon. \end{aligned}$$

Thus $x_j + y_j \to p + q$ as $j \to +\infty$.

On replacing y_j by $-y_j$ for all positive integers j, and using the result that $-y_j \to -q$ as $j \to +\infty$, we see that Thus $x_j - y_j \to p - q$ as $j \to +\infty$, as required. ∎

Lemma 1.5

Let $x_1, x_2, x_3, \ldots$ be a convergent infinite sequence of real numbers, and let c be a real number. Then

$$\lim_{j \to +\infty} (c x_j) = c \lim_{j \to +\infty} x_j.$$

Proof

Let some strictly positive real number ε be given. Then a strictly positive real number ε_1 can be chosen so that $|c|\,\varepsilon_1 \leq \varepsilon$. There then exists some positive integer N such that $|x_j - p| < \varepsilon_1$ whenever $j \geq N$, where $p = \lim_{j \to +\infty} x_j$. But then

$$|c x_j - c p| < |c|\,\varepsilon_1 \leq \varepsilon$$

whenever $j \geq N$. We conclude that $\lim_{j \to +\infty} c x_j = c p$, as required. ∎

Proposition 1.6

Let $x_1, x_2, x_3, \ldots$ and $y_1, y_2, y_3,$ be convergent infinite sequences of real numbers. Then the product of these sequences is convergent, and

$$\lim_{j \to +\infty} (x_j y_j) = \left(\lim_{j \to +\infty} x_j \right) \left(\lim_{j \to +\infty} y_j \right).$$

Proof

Let $u_j = x_j - p$ and $v_j = y_j - q$ for all positive integers j where $p = \lim_{j \to +\infty} x_j$ and $q = \lim_{j \to +\infty} y_j$. Then

$$\begin{aligned} \lim_{j \to +\infty} (u_j v_j) &= \lim_{j \to +\infty} (x_j y_j - x_j q - p y_j + pq) \\ &= \lim_{j \to +\infty} (x_j y_j) - q \lim_{j \to +\infty} x_j - p \lim_{j \to +\infty} y_j + pq \\ &= \lim_{j \to +\infty} (x_j y_j) - pq. \end{aligned}$$

Let some strictly positive real number ε be given. It follows from the definition of limits that $\lim\limits_{j \to +\infty} u_j = 0$ and $\lim\limits_{j \to +\infty} v_j = 0$. Therefore there exist positive integers N_1 and N_2 such that $|u_j| < \sqrt{\varepsilon}$ whenever $j \geq N_1$ and $|v_j| < \sqrt{\varepsilon}$ whenever $j \geq N_2$. Let N be the maximum of N_1 and N_2. If $j \geq N$ then $|u_j v_j| < \varepsilon$. Thus $\lim\limits_{j \to +\infty} u_j v_j = 0$, and therefore $\lim\limits_{j \to +\infty} (x_j y_j) - pq = 0$. The result follows.

Proposition 1.7

Let $x_1, x_2, x_3, \ldots$ and $y_1, y_2, y_3,$ be convergent infinite sequences of real numbers, where $y_j \neq 0$ for all positive integers j and $\lim\limits_{j \to +\infty} y_j \neq 0$. Then the quotient of the sequences (x_j) and (y_j) is convergent, and

$$\lim_{j \to +\infty} \frac{x_j}{y_j} = \frac{\lim\limits_{j \to +\infty} x_j}{\lim\limits_{j \to +\infty} y_j}.$$

Proof

Let $p = \lim_{j \to +\infty} x_j$ and Let $q = \lim_{j \to +\infty} y_j$. Then

$$\frac{x_j}{y_j} - \frac{p}{q} = \frac{qx_j - py_j}{qy_j}$$

for all positive integers j. Now there exists some positive integer N_1 such that $|y_j - q| < \frac{1}{2}|q|$ whenever $j \geq N_1$. Then $|y_j| \geq \frac{1}{2}|q|$ whenever $j \geq N_1$, and therefore

$$\left| \frac{x_j}{y_j} - \frac{p}{q} \right| \leq \frac{2}{|q|^2} |qx_j - py_j|$$

whenever $j \geq N_1$.

Let some strictly positive real number ε be given. Applying Lemma 1.5 and Proposition 1.4, we find that

$$\lim_{j\to+\infty}(qx_j - py_j) = q\lim_{j\to+\infty} x_j - p\lim_{j\to+\infty} y_j = qp - pq = 0.$$

Therefore there exists some positive integer N satisfying $N \geq N_1$ with the property that

$$|qx_j - py_j| < \tfrac{1}{2}|q|^2\varepsilon$$

whenever $j \geq N$. But then

$$\left|\frac{x_j}{y_j} - \frac{p}{q}\right| < \varepsilon$$

whenever $j \geq N$. Thus

$$\lim_{j\to+\infty}\frac{x_j}{y_j} = \frac{p}{q},$$

as required. ∎

1.9. Monotonic Sequences

An infinite sequence $x_1, x_2, x_3, \ldots$ of real numbers is said to be *strictly increasing* if $x_{j+1} > x_j$ for all positive integers j, *strictly decreasing* if $x_{j+1} < x_j$ for all positive integers j, *non-decreasing* if $x_{j+1} \geq x_j$ for all positive integers j, *non-increasing* if $x_{j+1} \leq x_j$ for all positive integers j. A sequence satisfying any one of these conditions is said to be *monotonic*; thus a monotonic sequence is either non-decreasing or non-increasing.

Theorem 1.8

Any non-decreasing sequence of real numbers that is bounded above is convergent. Similarly any non-increasing sequence of real numbers that is bounded below is convergent.

Proof

Let $x_1, x_2, x_3, \ldots$ be a non-decreasing sequence of real numbers that is bounded above. It follows from the Least Upper Bound Axiom that there exists a least upper bound p for the set $\{x_j : j \in \mathbb{N}\}$. We claim that the sequence converges to p.

Let some strictly positive real number ε be given. We must show that there exists some positive integer N such that $|x_j - p| < \varepsilon$ whenever $j \geq N$. Now $p - \varepsilon$ is not an upper bound for the set $\{x_j : j \in \mathbb{N}\}$ (since p is the least upper bound), and therefore there must exist some positive integer N such that $x_N > p - \varepsilon$. But then $p - \varepsilon < x_j \leq p$ whenever $j \geq N$, since the sequence is non-decreasing and bounded above by p. Thus $|x_j - p| < \varepsilon$ whenever $j \geq N$. Therefore $x_j \to p$ as $j \to +\infty$, as required.

If the sequence $x_1, x_2, x_3, \ldots$ is non-increasing and bounded below then the sequence $-x_1, -x_2, -x_3, \ldots$ is non-decreasing and bounded above, and is therefore convergent. It follows that the sequence $x_1, x_2, x_3, \ldots$ is also convergent. $\blacksquare$

1.10. Subsequences of Sequences of Real Numbers

Definition

Let $x_1, x_2, x_3, \ldots$ be an infinite sequence of real numbers. A *subsequence* of this infinite sequence is a sequence of the form $x_{j_1}, x_{j_2}, x_{j_3}, \ldots$ where $j_1, j_2, j_3, \ldots$ is an infinite sequence of positive integers with

$$j_1 < j_2 < j_3 < \cdots .$$

Let $x_1, x_2, x_3, \ldots$ be an infinite sequence of real numbers. The following sequences are examples of subsequences of the above sequence:—

$$x_1, x_3, x_5, x_7, \ldots$$

$$x_1, x_4, x_9, x_{16}, \ldots$$

Theorem 1.9 (Bolzano-Weierstrass)

Every bounded sequence of real numbers has a convergent subsequence.

Proof

Let $a_1, a_2, a_3, \ldots$ be a bounded sequence of real numbers. We define a *peak index* to be a positive integer j with the property that $a_j \geq a_k$ for all positive integers k satisfying $k \geq j$. Thus a positive integer j is a peak index if and only if the jth member of the infinite sequence $a_1, a_2, a_3, \ldots$ is greater than or equal to all succeeding members of the sequence. Let S be the set of all peak indices. Then

$$S = \{j \in \mathbb{N} : a_j \geq a_k \text{ for all } k \geq j\}.$$

First let us suppose that the set S of peak indices is infinite. Arrange the elements of S in increasing order so that $S = \{j_1, j_2, j_3, j_4, \ldots\}$, where $j_1 < j_2 < j_3 < j_4 < \cdots$. It follows from the definition of peak indices that $a_{j_1} \geq a_{j_2} \geq a_{j_3} \geq a_{j_4} \geq \cdots$. Thus $a_{j_1}, a_{j_2}, a_{j_3}, \ldots$ is a non-increasing subsequence of the original sequence $a_1, a_2, a_3, \ldots$. This subsequence is bounded below (since the original sequence is bounded). It follows from Theorem 1.8 that $a_{j_1}, a_{j_2}, a_{j_3}, \ldots$ is a convergent subsequence of the original sequence.

Now suppose that the set S of peak indices is finite. Choose a positive integer j_1 which is greater than every peak index. Then j_1 is not a peak index. Therefore there must exist some positive integer j_2 satisfying $j_2 > j_1$ such that $a_{j_2} > a_{j_1}$. Moreover j_2 is not a peak index (because j_2 is greater than j_1 and j_1 in turn is greater than every peak index). Therefore there must exist some positive integer j_3 satisfying $j_3 > j_2$ such that $a_{j_3} > a_{j_2}$. We can continue in this way to construct (by induction on j) a strictly increasing subsequence $a_{j_1}, a_{j_2}, a_{j_3}, \ldots$ of our original sequence. This increasing subsequence is bounded above (since the original sequence is bounded) and thus is convergent, by Theorem 1.8. This completes the proof of the Bolzano-Weierstrass Theorem. ∎