MAU23203—Analysis in Several Variables
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Section 8: Differentiation of Functions of One Real Variable

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8. Differentiation of Functions of One Real Variable

8.1. Interior Points and Open Sets in the Real Line

Definition

Let D be a subset of the set $\mathbb{R}$ of real numbers, and let s be a real number belonging to D. We say that s is an *interior point* of D if there exists some strictly positive number δ such that $x \in D$ for all real numbers x satisfying $s - \delta < x < s + \delta$. The *interior* of D is then the subset of D consisting of all real numbers belonging to D that are interior points of D.

It follows from the definition of open sets in Euclidean spaces that a subset D of the set $\mathbb{R}$ of real numbers is an open set in $\mathbb{R}$ if and only if every point of D is an interior point of D.
Let s be a real number. We say that a function $f\colon D \to \mathbb{R}$ is defined *around* s if the real number s is an interior point of the domain D of the function f. It follows that the function f is defined around s if and only if there exists some strictly positive real number δ such that $f(x)$ is defined for all real numbers x satisfying $s - \delta < x < s + \delta$.

8.2. Differentiable Functions of a Single Real Variable

We recall basic results of the theory of differentiable functions.

Definition

Let s be some real number, and let f be a real-valued function defined around s. The function f is said to be *differentiable* at s, with *derivative* $f'(s)$, if and only if the limit

$$f'(s) = \lim_{h \to 0} \frac{f(s+h) - f(s)}{h}$$

is well-defined. We denote by f', or by $\dfrac{df}{dx}$ the function whose value at s is the derivative $f'(s)$ of f at s.

Let s be some real number, and let f and g be real-valued functions defined around s that are differentiable at s. The basic rules of differential calculus then ensure that the functions $f+g$, $f-g$ and $f \cdot g$ are differentiable at s (where

$$(f+g)(x) = f(x) + g(x), \quad (f-g)(x) = f(x) - g(x)$$

and

$$(f.g)(x) = f(x)g(x)$$

for all real numbers x at which both $f(x)$ and $g(x)$ are defined), and

$$(f+g)'(s) = f'(s) + g'(s), \qquad (f-g)'(s) = f'(s) - g'(s).$$

$$(f \cdot g)'(s) = f'(s)g(s) + f(s)g'(s) \quad (\textit{Product Rule}).$$

If moreover $g(s) \neq 0$ then the function f/g is differentiable at s (where $(f/g)(x) = f(x)/g(x)$ where both $f(x)$ and $g(x)$ are defined), and

$$(f/g)'(s) = \frac{f'(s)g(s) - f(s)g'(s)}{g(s)^2} \quad (\textit{Quotient Rule}).$$

Moreover if h is a real-valued function defined around $f(s)$ which is differentiable at $f(s)$ then the composition function $h \circ f$ is differentiable at $f(s)$ and

$$(h \circ f)'(s) = h'(f(s))f'(s) \quad (\textit{Chain Rule}).$$

Derivatives of some standard functions are as follows:—

$$\frac{d}{dx}(x^m) = mx^{m-1}, \quad \frac{d}{dx}(\sin x) = \cos x, \quad \frac{d}{dx}(\cos x) = -\sin x,$$

$$\frac{d}{dx}(\exp x) = \exp x, \quad \frac{d}{dx}(\log x) = \frac{1}{x} \ \ (x > 0).$$

8.3. Rolle's Theorem

Theorem 8.1 (Rolle's Theorem)

Let $f\colon [a,b] \to \mathbb{R}$ be a real-valued function defined on some interval $[a,b]$. Suppose that f is continuous on $[a,b]$ and is differentiable on (a,b). Suppose also that $f(a) = f(b)$. Then there exists some real number s satisfying $a < s < b$ which has the property that $f'(s) = 0$.

Proof

First we show that if the function f attains its minimum value at u, and if $a < u < b$, then $f'(u) = 0$. Now the difference quotient

$$\frac{f(u+h) - f(u)}{h}$$

is non-negative for all sufficiently small positive values of h; therefore $f'(u) \geq 0$. On the other hand, this difference quotient is non-positive for all sufficiently small negative values of h; therefore $f'(u) \leq 0$. We deduce therefore that $f'(u) = 0$.

Similarly if the function f attains its maximum value at v, and if $a < v < b$, then $f'(v) = 0$. (Indeed the result for local maxima can be deduced from the corresponding result for local minima simply by replacing the function f by $-f$.)

Now the function f is continuous on the closed bounded interval $[a, b]$. It therefore follows from the Extreme Value Theorem that there must exist real numbers u and v in the interval $[a, b]$ with the property that $f(u) \leq f(x) \leq f(v)$ for all real numbers x satisfying $a \leq x \leq b$ (see Theorem 4.21). If $a < u < b$ then $f'(u) = 0$ and we can take $s = u$. Similarly if $a < v < b$ then $f'(v) = 0$ and we can take $s = v$. The only remaining case to consider is when both u and v are endpoints of the interval $[a, b]$. In that case the function f is constant on $[a, b]$, since $f(a) = f(b)$, and we can choose s to be any real number satisfying $a < s < b$. ∎

8.4. The Mean Value Theorem

Rolle's Theorem can be generalized to yield the following important theorem.

Theorem 8.2 (The Mean Value Theorem)

Let $f\colon [a,b] \to \mathbb{R}$ be a real-valued function defined on some interval $[a,b]$. Suppose that f is continuous on $[a,b]$ and is differentiable on (a,b). Then there exists some real number s satisfying $a < s < b$ which has the property that

$$f(b) - f(a) = f'(s)(b-a).$$

Proof

Let $g\colon [a,b] \to \mathbb{R}$ be the real-valued function on the closed interval $[a,b]$ defined by

$$g(x) = f(x) - \frac{b-x}{b-a} f(a) - \frac{x-a}{b-a} f(b).$$

Then the function g is continuous on $[a,b]$ and differentiable on (a,b). Moreover $g(a) = 0$ and $g(b) = 0$. It follows from Rolle's Theorem (Theorem 8.1) that $g'(s) = 0$ for some real number s satisfying $a < s < b$. But

$$g'(s) = f'(s) - \frac{f(b) - f(a)}{b-a}.$$

Therefore $f(b) - f(a) = f'(s)(b-a)$, as required. ∎

A number of basic principles of single variable calculus follow as immediate consequences of the Mean Value Theorem (Theorem 8.2). A number of such consequences are presented in the following corollaries.

Corollary 8.3

Let $f\colon [a, b] \to \mathbb{R}$ be a real-valued function defined on some interval $[a, b]$. Suppose that f is continuous on $[a, b]$ and is differentiable on (a, b) and that $f'(x) > 0$ for all real numbers x satisfying $a < x < b$. Then $f(b) > f(a)$.

Corollary 8.4

Let $f\colon [a,b] \to \mathbb{R}$ be a real-valued function defined on some interval $[a,b]$. Suppose that f is continuous on $[a,b]$ and is differentiable on (a,b) and that $f'(x) = 0$ for all real numbers x satisfying $a < x < b$. Then $f(x) = f(a)$ for all $x \in [a,b]$.

Corollary 8.5

Let $f\colon [a,b] \to \mathbb{R}$ be a real-valued function defined on some interval $[a,b]$, and let M be a real number. Suppose that f is continuous on $[a,b]$ and is differentiable on (a,b) and that $f'(x) \leq M$ for all real numbers x satisfying $a < x < b$. Then $f(x) \leq f(a) + M(x-a)$ for all $x \in [a,b]$.

Corollary 8.6

Let $f\colon [a,b] \to \mathbb{R}$ be a real-valued function defined on some interval $[a,b]$, and let M be a real number. Suppose that f is continuous on $[a,b]$ and is differentiable on (a,b) and that $|f'(x)| \leq M$ for all real numbers x satisfying $a < x < b$. Then $|f(b) - f(a)| \leq M(b-a)$.

8.5. Concavity and the Second Derivative

Proposition 8.7

Let s and h be real numbers, and let f be a twice differentiable real-valued function defined on some open interval containing s and $s+h$. Then there exists a real number θ satisfying $0 < \theta < 1$ for which

$$f(s+h) = f(s) + hf'(s) + \tfrac{1}{2}h^2 f''(s+\theta h).$$

Proof

Let I be an open interval, containing the real numbers 0 and 1, chosen to ensure that $f(s + th)$ is defined for all $t \in I$, and let $q\colon I \to \mathbb{R}$ be defined so that

$$q(t) = f(s + th) - f(s) - thf'(s) - t^2(f(s + h) - f(s) - hf'(s)).$$

for all $t \in I$. Differentiating, we find that

$$q'(t) = hf'(s + th) - hf'(s) - 2t(f(s + h) - f(s) - hf'(s))$$

and

$$q''(t) = h^2 f''(s + th) - 2(f(s + h) - f(s) - hf'(s)).$$

Now $q(0) = q(1) = 0$. It follows from Rolle's Theorem, applied to the function q on the interval $[0, 1]$, that there exists some real number φ satisfying $0 < \varphi < 1$ for which $q'(\varphi) = 0$.

Then $q'(0) = q'(\varphi) = 0$, and therefore Rolle's Theorem can be applied to the function q' on the interval $[0, \varphi]$ to prove the existence of some real number θ satisfying $0 < \theta < \varphi$ for which $q''(\theta) = 0$. Then

$$0 = q''(\theta) = h^2 f''(s + \theta h) - 2(f(s + h) - f(s) - hf'(s)).$$

Rearranging, we find that

$$f(s + h) = f(s) + hf'(s) + \tfrac{1}{2} h^2 f''(s + \theta h),$$

as required. $\blacksquare$

Corollary 8.8

Let $f\colon (s-\delta_0, s+\delta_0)$ be a twice-differentiable function throughout some open interval $(s-\delta_0, s+\delta_0)$ centred on a real number s. Suppose that $f''(s+h) > 0$ for all real numbers h satisfying $|h| < \delta_0$. Then

$$f(s+h) \geq f(s) + hf'(s)$$

for all real numbers h satisfying $|h| < \delta_0$.

It follows from Corollary 8.8 that if a twice-differentiable function has positive second derivative throughout some open interval, then it is concave upwards throughout that interval. In particular the function has a local minimum at any point of that open interval where the first derivative is zero and the second derivative is positive.

Corollary 8.9

Let $f\colon D \to \mathbb{R}$ be a twice-differentiable function defined over a subset D of $\mathbb{R}$, and let s be a point in the interior of D. Suppose that $f'(s) = 0$ and that $f''(x) > 0$ for all real numbers x belonging to some sufficiently small neighbourhood of x. Then s is a local minimum for the function f.

8.6. Taylor's Theorem

The result obtained in Proposition 8.7 is a special case of a more general result. That more general result is a version of Taylor's Theorem with remainder. The proof of this theorem will make use of the following lemma.

Lemma 8.10

Let s and h be real numbers, let f be a k times differentiable real-valued function defined on some open interval containing s and $s+h$, let $c_0, c_1, \ldots, c_{k-1}$ be real numbers, and let

$$p(t) = f(s+th) - \sum_{n=0}^{k-1} c_n t^n.$$

for all real numbers t belonging to some open interval D for which $0 \in D$ and $1 \in D$. Then $p^{(n)}(0) = 0$ for all integers n satisfying $0 \leq n < k$ if and only if

$$c_n = \frac{h^n f^{(n)}(s)}{n!}$$

for all integers n satisfying $0 \leq n < k$.

Proof

On setting $t = 0$, we find that $p(0) = f(s) - c_0$, and thus $p(0) = 0$ if and only if $c_0 = f(s)$.

Let the integer n satisfy $0 < n < k$. On differentiating $p(t)$ n times with respect to t, we find that

$$p^{(n)}(t) = h^n f^{(n)}(s + th) - \sum_{j=n}^{k-1} \frac{j!}{(j-n)!} c_j t^{j-n}.$$

Then, on setting $t = 0$, we find that only the term with $j = n$ contributes to the value of the sum on the right hand side of the above identity, and therefore

$$p^{(n)}(0) = h^n f^{(n)}(s) - n! c_n.$$

The result follows. ∎

Theorem 8.11

(Taylor's Theorem) Let s and h be real numbers, and let f be a k times differentiable real-valued function defined on some open interval containing s and $s + h$. Then

$$f(s+h) = f(s) + \sum_{n=1}^{k-1} \frac{h^n}{n!} f^{(n)}(s) + \frac{h^k}{k!} f^{(k)}(s + \theta h)$$

for some real number θ satisfying $0 < \theta < 1$.

Proof

Let D be an open interval, containing the real numbers 0 and 1, chosen to ensure that $f(s+th)$ is defined for all $t \in D$, and let $p: D \to \mathbb{R}$ be defined so that

$$p(t) = f(s+th) - f(s) - \sum_{n=1}^{k-1} \frac{t^n h^n}{n!} f^{(n)}(s)$$

for all $t \in D$. A straightforward calculation shows that $p^{(n)}(0) = 0$ for $n = 0, 1, \ldots, k-1$ (see Lemma 8.10). Thus if $q(t) = p(t) - p(1)t^k$ for all $s \in [0,1]$ then $q^{(n)}(0) = 0$ for $n = 0, 1, \ldots, k-1$, and $q(1) = 0$. We can therefore apply Rolle's Theorem (Theorem 8.1) to the function q on the interval $[0,1]$ to deduce the existence of some real number t_1 satisfying $0 < t_1 < 1$ for which $q'(t_1) = 0$. We can then apply Rolle's Theorem to the function q' on the interval $[0, t_1]$ to deduce the existence of some real number t_2 satisfying $0 < t_2 < t_1$ for which $q''(t_2) = 0$.

By continuing in this fashion, applying Rolle's Theorem in turn to the functions $q'', q''', \ldots, q^{(k-1)}$, we deduce the existence of real numbers $t_1, t_2, \ldots, t_k$ satisfying $0 < t_k < t_{k-1} < \cdots < t_1 < 1$ with the property that $q^{(n)}(t_n) = 0$ for $n = 1, 2, \ldots, k$. Let $\theta = t_k$. Then $0 < \theta < 1$ and

$$0 = \frac{1}{k!} q^{(k)}(\theta) = \frac{1}{k!} p^{(k)}(\theta) - p(1) = \frac{h^k}{k!} f^{(k)}(s + \theta h) - p(1),$$

hence

$$\begin{aligned} f(s+h) &= f(s) + \sum_{n=1}^{k-1} \frac{h^n}{n!} f^{(n)}(s) + p(1) \\ &= f(s) + \sum_{n=1}^{k-1} \frac{h^n}{n!} f^{(n)}(s) + \frac{h^k}{k!} f^{(k)}(s + \theta h), \end{aligned}$$

as required. ∎

Corollary 8.12

Let $f\colon D \to \mathbb{R}$ be a k-times continuously differentiable function defined over an open subset D of $\mathbb{R}$ and let $s \in \mathbb{R}$. Then given any strictly positive real number ε, there exists some strictly positive real number δ such that

$$\left| f(s+h) - f(s) - \sum_{n=1}^{k} \frac{h^n}{n!} f^{(n)}(s) \right| < \varepsilon |h|^k$$

whenever $|h| < \delta$.

Proof

The function f is k-times continuously differentiable, and therefore its kth derivative $f^{(k)}$ is continuous. Let some strictly positive real number ε be given. Then there exists some strictly positive real number δ that is small enough to ensure that $s + h \in D$ and $|f^{(k)}(s + h) - f^{(k)}(s)| < k!\varepsilon$ whenever $|h| < \delta$. If h is an real number satisfying $|h| < \delta$, and if θ is a real number satisfying $0 < \theta < 1$, then $s + \theta h \in D$ and $|f^{(k)}(s + \theta h) - f^{(k)}(s)| < k!\varepsilon$. Now it follows from Taylor's Theorem (Theorem 8.11) that, given any real number h satisfying $|h| < \delta$ there exists some real number θ satisfying $0 < \theta < 1$ for which

$$f(s + h) = f(s) + \sum_{n=1}^{k-1} \frac{h^n}{n!} f^{(n)}(s) + \frac{h^k}{k!} f^{(k)}(s + \theta h).$$

Then

$$\begin{aligned}\left| f(s+h) - f(s) - \sum_{n=1}^{k} \frac{h^n}{n!} f^{(n)}(s) \right| \\ &= \frac{|h|^k}{k!} |f^{(k)}(s+\theta h) - f^{(k)}(s)| \\ &< \varepsilon |h|^k,\end{aligned}$$

as required. ∎

Let $f\colon [a,b] \to \mathbb{R}$ be a continuous function on a closed interval $[a,b]$. We say that f is *continuously differentiable* on $[a,b]$ if the derivative $f'(x)$ of f exists for all x satisfying $a < x < b$, the one-sided derivatives

$$\begin{aligned} f'(a) &= \lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h}, \\ f'(b) &= \lim_{h \to 0^-} \frac{f(b+h) - f(b)}{h} \end{aligned}$$

exist at the endpoints of $[a,b]$, and the function f' is continuous on $[a,b]$.

If $f\colon [a,b] \to \mathbb{R}$ is continuous, and if $F(x) = \int_a^x f(t)\,dt$ for all $x \in [a,b]$ then the one-sided derivatives of F at the endpoints of $[a,b]$ exist, and

$$\lim_{h \to 0^+} \frac{F(a+h) - F(a)}{h} = f(a), \qquad \lim_{h \to 0^-} \frac{F(b+h) - F(b)}{h} = f(b).$$

One can verify these results by adapting the proof of the Fundamental Theorem of Calculus.

Proposition 8.13

Let f be a continuously differentiable real-valued function on the interval $[a, b]$. Then

$$\int_a^b \frac{df(x)}{dx}\, dx = f(b) - f(a)$$

Proof

Define $g\colon [a, b] \to \mathbb{R}$ by

$$g(x) = f(x) - f(a) - \int_a^x \frac{df(t)}{dt}\, dt.$$

Then $g(a) = 0$, and

$$\frac{dg(x)}{dx} = \frac{df(x)}{dx} - \frac{d}{dx}\left(\int_a^x \frac{df(t)}{dt}\, dt\right) = 0$$

for all x satisfying $a < x < b$, by the Fundamental Theorem of Calculus. Now it follows from the Mean Value Theorem (Theorem 8.2) that there exists some s satisfying $a < s < b$ for which $g(b) - g(a) = (b - a)g'(s)$. We deduce therefore that $g(b) = 0$, which yields the required result. ∎

Corollary 8.14 (Integration by Parts)

Let f and g be continuously differentiable real-valued functions on the interval $[a, b]$. Then

$$\int_a^b f(x)\,\frac{dg(x)}{dx}\,dx = f(b)g(b) - f(a)g(a) - \int_a^b g(x)\,\frac{df(x)}{dx}\,dx.$$

Proof

This result follows from Proposition 8.13 on integrating the identity

$$f(x)\frac{dg(x)}{dx} = \frac{d}{dx}\left(f(x)g(x)\right) - g(x)\,\frac{df(x)}{dx}. \qquad \blacksquare$$

Corollary 8.15 (Integration by Substitution)

Let $u\colon [a,b] \to \mathbb{R}$ be a continuously differentiable monotonically increasing function on the interval $[a,b]$, and let $c = u(a)$ and $d = u(b)$. Then

$$\int_c^d f(x)\,dx = \int_a^b f(u(t))\frac{du(t)}{dt}\,dt.$$

for all continuous real-valued functions f on $[c,d]$.

Proof

Let F and G be the functions on $[a, b]$ defined by

$$F(x) = \int_c^{u(x)} f(y)dy, \qquad G(x) = \int_a^x f(u(t))\frac{du(t)}{dt}dt.$$

Then $F(a) = 0 = G(a)$. Moreover $F(x) = H(u(x))$, where

$$H(s) = \int_c^s f(y)\,dy,$$

and $H'(s) = f(s)$ for all $s \in [a, b]$. Using the Chain Rule and the Fundamental Theorem of Calculus, we deduce that

$$F'(x) = H'(u(x))u'(x) = f(u(x))u'(x) = G'(x)$$

for all $x \in (a, b)$. On applying the Mean Value Theorem (Theorem 8.2) to the function $F - G$ on the interval $[a, b]$, we see that $F(b) - G(b) = F(a) - G(a) = 0$. Thus $H(d) = F(b) = G(b)$, which yields the required identity. ∎

Proposition 8.16 (Taylor's Theorem with Integral Remainder)

Let s and h be real numbers, and let f be a function whose first k derivatives are continuous on an interval containing s and $s+h$. Then

$$\begin{aligned} f(s+h) &= f(s) + \sum_{n=1}^{k-1} \frac{h^n}{n!} f^{(n)}(s) \\ &\quad + \frac{h^k}{(k-1)!} \int_0^1 (1-t)^{k-1} f^{(k)}(s+th)\,dt. \end{aligned}$$

Proof

Let

$$r_m(s,h) = \frac{h^m}{(m-1)!} \int_0^1 (1-t)^{m-1} f^{(m)}(s+th)\,dt$$

for $m = 1, 2, \ldots, k-1$. Then

$$r_1(s,h) = h \int_0^1 f'(s+th)\,dt = \int_0^1 \frac{d}{dt} f(s+th)\,dt = f(s+h) - f(s).$$

Let m be an integer between 1 and $k-2$. It follows from the rule for Integration by Parts (Corollary 8.14) that

$$\begin{aligned}
r_{m+1}(s,h) &= \frac{h^{m+1}}{m!}\int_0^1 (1-t)^m f^{(m+1)}(s+th)\,dt \\
&= \frac{h^m}{m!}\int_0^1 (1-t)^m \frac{d}{dt}\left(f^{(m)}(s+th)\right)\,dt \\
&= \frac{h^m}{m!}\left[(1-t)^m f^{(m)}(s+th)\right]_0^1 \\
&\qquad - \frac{h^m}{m!}\int_0^1 \frac{d}{dt}\left((1-t)^m\right) f^{(m)}(s+th)\,dt \\
&= -\frac{h^m}{m!} f^{(m)}(s) \\
&\qquad + \frac{h^m}{(m-1)!}\int_0^1 (1-t)^{m-1} f^{(m)}(s+th)\,dt \\
&= r_m(s,h) - \frac{h^m}{m!} f^{(m)}(s).
\end{aligned}$$

Thus

$$r_m(s, h) = \frac{h^m}{m!} f^{(m)}(s) + r_{m+1}(s, h)$$

for $m = 1, 2, \ldots, k-1$. It follows by induction on k that

$$\begin{aligned} f(s+h) &= f(s) + \sum_{n=1}^{k-1} \frac{h^n}{n!} f^{(n)}(s) + r_k(s, h) \\ &= f(s) + \sum_{n=1}^{k-1} \frac{h^n}{n!} f^{(n)}(s) \\ &\qquad + \frac{h^k}{(k-1)!} \int_0^1 (1-t)^{k-1} f^{(k)}(s+th)\, dt, \end{aligned}$$

as required. ∎