MAU23203—Analysis in Several Variables
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Section 3: Open and Closed Sets in Euclidean Spaces

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3. Open and Closed Sets in Euclidean Spaces

3.1. Open Sets in Euclidean Spaces

Definition

Given a point $\mathbf{p}$ of $\mathbb{R}^n$ and a non-negative real number r, the *open ball* $B(\mathbf{p}, r)$ in $\mathbb{R}^n$ of *radius* r about $\mathbf{p}$ is defined to be the subset of $\mathbb{R}^n$ defined so that

$$B(\mathbf{p}, r) = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x} - \mathbf{p}| < r\}.$$

(Thus $B(\mathbf{p}, r)$ is the set consisting of all points of $\mathbb{R}^n$ that lie within a sphere of radius r centred on the point $\mathbf{p}$.)

The *open ball* $B(\mathbf{p}, r)$ of radius r about a point $\mathbf{p}$ of $\mathbb{R}^n$ is bounded by the *sphere* of radius r about $\mathbf{p}$. This sphere is the set

$$\{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x} - \mathbf{p}| = r\}.$$

Definition

A subset V of $\mathbb{R}^n$ is said to be an *open set* (in $\mathbb{R}^n$) if, given any point $\mathbf{p}$ of V, there exists some strictly positive real number δ such that $B(\mathbf{p}, \delta) \subset V$, where $B(\mathbf{p}, \delta)$ is the open ball in $\mathbb{R}^n$ of radius δ about the point $\mathbf{p}$, defined so that

$$B(\mathbf{p}, \delta) = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x} - \mathbf{p}| < \delta\}.$$

Example

Let $H = \{(x, y, z) \in \mathbb{R}^3 : z > c\}$, where c is some real number. Then H is an open set in $\mathbb{R}^3$. Indeed let $\mathbf{p}$ be a point of H. Then $\mathbf{p} = (u, v, w)$, where $w > c$. Let $\delta = w - c$. If the distance from a point (x, y, z) to the point (u, v, w) is less than δ then $|z - w| < \delta$, and hence $z > c$, so that $(x, y, z) \in H$. Thus $B(\mathbf{p}, \delta) \subset H$, and therefore H is an open set.

The previous example can be generalized. Given any integer i between 1 and n, and given any real number c_i, the sets

$$\{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_i > c_i\}$$

and

$$\{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_i < c_i\}$$

are open sets in $\mathbb{R}^n$.

Example

Let

$$V = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 < 9\}.$$

Then the subset V of $\mathbb{R}^3$ is the open ball of radius 3 in $\mathbb{R}^3$ about the origin. This open ball is an open set. Indeed let $\mathbf{x}$ be a point of V. Then $|\mathbf{x}| < 3$. Let $\delta = 3 - |\mathbf{x}|$. Then $\delta > 0$. Moreover if $\mathbf{y}$ is a point of $\mathbb{R}^3$ that satisfies $|\mathbf{y} - \mathbf{x}| < \delta$ then

$$|\mathbf{y}| = |\mathbf{x} + (\mathbf{y} - \mathbf{x})| \leq |\mathbf{x}| + |\mathbf{y} - \mathbf{x}| < |\mathbf{x}| + \delta = 3,$$

and therefore $\mathbf{y} \in V$. This proves that V is an open set.

More generally, an open ball of any positive radius about any point of a Euclidean space $\mathbb{R}^n$ of any dimension n is an open set in that Euclidean space. A more general result is proved below (see Lemma 3.1).

3.2. Open Sets in Subsets of Euclidean Spaces

Definition

Let X be a subset of $\mathbb{R}^n$. Given a point $\mathbf{p}$ of X and a non-negative real number r, the *open ball* $B_X(\mathbf{p}, r)$ in X of *radius* r about $\mathbf{p}$ is defined to be the subset of X defined so that

$$B_X(\mathbf{p}, r) = \{\mathbf{x} \in X : |\mathbf{x} - \mathbf{p}| < r\}.$$

(Thus $B_X(\mathbf{p}, r)$ is the set consisting of all points of X that lie within a sphere of radius r centred on the point $\mathbf{p}$.)

Definition

Let X be a subset of $\mathbb{R}^n$. A subset V of X is said to be *open* in X if, given any point $\mathbf{p}$ of V, there exists some strictly positive real number δ such that $B_X(\mathbf{p}, \delta) \subset V$, where $B_X(\mathbf{p}, \delta)$ is the open ball in X of radius δ about on the point $\mathbf{p}$. The empty set $\emptyset$ is also defined to be an open set in X.

Example

Let U be an open set in $\mathbb{R}^n$. Then for any subset X of $\mathbb{R}^n$, the intersection $U \cap X$ is open in X. (This follows directly from the definitions.) Thus for example, let S^2 be the unit sphere in $\mathbb{R}^3$, given by

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$$

and let N be the subset of S^2 given by

$$N = \{(x, y, z) \in \mathbb{R}^n : x^2 + y^2 + z^2 = 1 \text{ and } z > 0\}.$$

Then N is open in S^2, since $N = H \cap S^2$, where H is the open set in $\mathbb{R}^3$ given by

$$H = \{(x, y, z) \in \mathbb{R}^3 : z > 0\}.$$

Note that N is not itself an open set in $\mathbb{R}^3$. Indeed the point $(0, 0, 1)$ belongs to N, but, for any $\delta > 0$, the open ball (in $\mathbb{R}^3$) of radius δ about $(0, 0, 1)$ contains points (x, y, z) for which $x^2 + y^2 + z^2 \neq 1$. Thus the open ball of radius δ about the point $(0, 0, 1)$ is not a subset of N.

Lemma 3.1

Let X be a subset of $\mathbb{R}^n$, and let $\mathbf{p}$ be a point of X. Then, for any positive real number r, the open ball $B_X(\mathbf{p}, r)$ in X of radius r about $\mathbf{p}$ is open in X.

Proof

Let $\mathbf{x}$ be an element of $B_X(\mathbf{p}, r)$. We must show that there exists some $\delta > 0$ such that $B_X(\mathbf{x}, \delta) \subset B_X(\mathbf{p}, r)$. Let $\delta = r - |\mathbf{x} - \mathbf{p}|$. Then $\delta > 0$, since $|\mathbf{x} - \mathbf{p}| < r$. Moreover if $\mathbf{y} \in B_X(\mathbf{x}, \delta)$ then

$$|\mathbf{y} - \mathbf{p}| \leq |\mathbf{y} - \mathbf{x}| + |\mathbf{x} - \mathbf{p}| < \delta + |\mathbf{x} - \mathbf{p}| = r,$$

by the Triangle Inequality, and hence $\mathbf{y} \in B_X(\mathbf{p}, r)$. Thus $B_X(\mathbf{x}, \delta) \subset B_X(\mathbf{p}, r)$. This shows that $B_X(\mathbf{p}, r)$ is an open set, as required. ∎

Lemma 3.2

Let X be a subset of $\mathbb{R}^n$, and let $\mathbf{p}$ be a point of X. Then, for any non-negative real number r, the set $\{\mathbf{x} \in X : |\mathbf{x} - \mathbf{p}| > r\}$ is an open set in X.

Proof

Let $\mathbf{x}$ be a point of X satisfying $|\mathbf{x} - \mathbf{p}| > r$, and let $\mathbf{y}$ be any point of X satisfying $|\mathbf{y} - \mathbf{x}| < \delta$, where $\delta = |\mathbf{x} - \mathbf{p}| - r$. Then

$$|\mathbf{x} - \mathbf{p}| \leq |\mathbf{x} - \mathbf{y}| + |\mathbf{y} - \mathbf{p}|,$$

by the Triangle Inequality, and therefore

$$|\mathbf{y} - \mathbf{p}| \geq |\mathbf{x} - \mathbf{p}| - |\mathbf{y} - \mathbf{x}| > |\mathbf{x} - \mathbf{p}| - \delta = r.$$

Thus $B_X(\mathbf{x}, \delta)$ is contained in the given set. The result follows. ∎

Proposition 3.3

Let X be a subset of $\mathbb{R}^n$. The collection of open sets in X has the following properties:—

(i) *the empty set $\emptyset$ and the whole set X are both open in X;*

(ii) *the union of any collection of open sets in X is itself open in X;*

(iii) *the intersection of any finite collection of open sets in X is itself open in X.*

Proof

The empty set $\emptyset$ is an open set by convention. Moreover the definition of an open set is satisfied trivially by the whole set X. This proves (i).

Let $\mathcal{A}$ be any collection of open sets in X, and let U denote the union of all the open sets belonging to $\mathcal{A}$. We must show that U is itself open in X. Let $\mathbf{x} \in U$. Then $\mathbf{x} \in V$ for some set V belonging to the collection $\mathcal{A}$. It follows that there exists some $\delta > 0$ such that $B_X(\mathbf{x}, \delta) \subset V$. But $V \subset U$, and thus $B_X(\mathbf{x}, \delta) \subset U$. This shows that U is open in X. This proves (ii).

Finally let $V_1, V_2, V_3, \ldots, V_k$ be a *finite* collection of subsets of X that are open in X, and let V denote the intersection $V_1 \cap V_2 \cap \cdots \cap V_k$ of these sets. Let $\mathbf{x} \in V$. Now $\mathbf{x} \in V_j$ for $j = 1, 2, \ldots, k$, and therefore there exist strictly positive real numbers $\delta_1, \delta_2, \ldots, \delta_k$ such that $B_X(\mathbf{x}, \delta_j) \subset V_j$ for $j = 1, 2, \ldots, k$. Let δ be the minimum of $\delta_1, \delta_2, \ldots, \delta_k$. Then $\delta > 0$. (This is where we need the fact that we are dealing with a finite collection of sets.) Now $B_X(\mathbf{x}, \delta) \subset B_X(\mathbf{x}, \delta_j) \subset V_j$ for $j = 1, 2, \ldots, k$, and thus $B_X(\mathbf{x}, \delta) \subset V$. Thus the intersection V of the sets $V_1, V_2, \ldots, V_k$ is itself open in X. This proves (iii). $\blacksquare$

Example

The set $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 < 4 \text{ and } z > 1\}$ is an open set in $\mathbb{R}^3$, since it is the intersection of the open ball of radius 2 about the origin with the open set $\{(x, y, z) \in \mathbb{R}^3 : z > 1\}$.

Example

The set $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 < 4 \text{ or } z > 1\}$ is an open set in $\mathbb{R}^3$, since it is the union of the open ball of radius 2 about the origin with the open set $\{(x, y, z) \in \mathbb{R}^3 : z > 1\}$.

Example

The set

$$\{(x, y, z) \in \mathbb{R}^3 : (x-n)^2 + y^2 + z^2 < \tfrac{1}{4} \text{ for some } n \in \mathbb{Z}\}$$

is an open set in $\mathbb{R}^3$, since it is the union of the open balls of radius $\frac{1}{2}$ about the points $(n, 0, 0)$ for all integers n.

Example

For each positive integer k, let

$$V_k = \{(x, y, z) \in \mathbb{R}^3 : k^2(x^2 + y^2 + z^2) < 1\}.$$

Now each set V_k is an open ball of radius $1/k$ about the origin, and is therefore an open set in $\mathbb{R}^3$. However the intersection of the sets V_k for all positive integers k is the set $\{(0, 0, 0)\}$, and thus the intersection of the sets V_k for all positive integers k is not itself an open set in $\mathbb{R}^3$. This example demonstrates that infinite intersections of open sets need not be open.

Proposition 3.4

Let X be a subset of $\mathbb{R}^n$, and let U be a subset of X. Then U is open in X if and only if there exists some open set V in $\mathbb{R}^n$ for which $U = V \cap X$.

Proof

First suppose that $U = V \cap X$ for some open set V in $\mathbb{R}^n$. Let $\mathbf{u} \in U$. Then the definition of open sets in $\mathbb{R}^n$ ensures that there exists some positive real number δ such that

$$\{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x} - \mathbf{u}| < \delta\} \subset V.$$

Then

$$\{\mathbf{x} \in X : |\mathbf{x} - \mathbf{u}| < \delta\} \subset U.$$

This shows that U is open in X.

Conversely suppose that the subset U of X is open in X. For each point $\mathbf{u}$ of U there exists some positive real number $\delta_{\mathbf{u}}$ such that

$$\{\mathbf{x} \in X : |\mathbf{x} - \mathbf{u}| < \delta_{\mathbf{u}}\} \subset U.$$

For each $\mathbf{u} \in U$, let $B(\mathbf{u}, \delta_{\mathbf{u}})$ denote the open ball in $\mathbb{R}^n$ of radius $\delta_{\mathbf{u}}$ about the point $\mathbf{u}$, so that

$$B(\mathbf{u}, \delta_{\mathbf{u}}) = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x} - \mathbf{u}| < \delta_{\mathbf{u}}\}$$

for all $\mathbf{u} \in U$, and let V be the union of all the open balls $B(\mathbf{u}, \delta_{\mathbf{u}})$ as $\mathbf{u}$ ranges over all the points of U. Then V is an open set in $\mathbb{R}^n$.

Indeed every open ball in $\mathbb{R}^n$ is an open set (Lemma 3.1), and any union of open sets in $\mathbb{R}^n$ is itself an open set (Proposition 3.3). The set V is a union of open balls. It is therefore a union of open sets, and so must itself be an open set.

Now $B(\mathbf{u}, \delta_{\mathbf{u}}) \cap X \subset U$. for all $\mathbf{u} \in U$. Also every point of V belongs to $B(\mathbf{u}, \delta_{\mathbf{u}})$ for at least one point $\mathbf{u}$ of U. It follows that $V \cap X \subset U$. But $\mathbf{u} \in B(\mathbf{u}, \delta_{\mathbf{u}})$ and $B(\mathbf{u}, \delta_{\mathbf{u}}) \subset V$ for all $\mathbf{u} \in U$, and therefore $U \subset V$, and thus $U \subset V \cap X$. It follows that $U = V \cap X$, as required. ∎

3.3. Convergence of Sequences and Open Sets

Lemma 3.5

A sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ of points in $\mathbb{R}^n$ converges to a point $\mathbf{p}$ if and only if, given any open set U which contains $\mathbf{p}$, there exists some positive integer N such that $\mathbf{x}_j \in U$ for all j satisfying $j \geq N$.

Proof

Suppose that the sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ has the property that, given any open set U which contains $\mathbf{p}$, there exists some positive integer N such that $\mathbf{x}_j \in U$ whenever $j \geq N$. Let $\varepsilon > 0$ be given. The open ball $B(\mathbf{p}, \varepsilon)$ of radius ε about $\mathbf{p}$ is an open set by Lemma 3.1. Therefore there exists some positive integer N such that $\mathbf{x}_j \in B(\mathbf{p}, \varepsilon)$ whenever $j \geq N$. Thus $|\mathbf{x}_j - \mathbf{p}| < \varepsilon$ whenever $j \geq N$. This shows that the sequence converges to $\mathbf{p}$.

Conversely, suppose that the sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ converges to $\mathbf{p}$. Let U be an open set which contains $\mathbf{p}$. Then there exists some $\varepsilon > 0$ such that the open ball $B(\mathbf{p}, \varepsilon)$ of radius ε about $\mathbf{p}$ is a subset of U. Thus there exists some $\varepsilon > 0$ such that U contains all points $\mathbf{x}$ of $\mathbb{R}^n$ that satisfy $|\mathbf{x} - \mathbf{p}| < \varepsilon$. But there exists some positive integer N with the property that $|\mathbf{x}_j - \mathbf{p}| < \varepsilon$ whenever $j \geq N$, since the sequence converges to $\mathbf{p}$. Therefore $\mathbf{x}_j \in U$ whenever $j \geq N$, as required. $\blacksquare$

3.4. Closed Sets in Euclidean Spaces

Let X be a subset of $\mathbb{R}^n$. A subset F of X is said to be *closed* in X if and only if its complement $X \setminus F$ in X is open in X. (Recall that $X \setminus F = \{\mathbf{x} \in X : \mathbf{x} \notin F\}$.)

Example

The sets $\{(x, y, z) \in \mathbb{R}^3 : z \geq c\}$, $\{(x, y, z) \in \mathbb{R}^3 : z \leq c\}$, and $\{(x, y, z) \in \mathbb{R}^3 : z = c\}$ are closed sets in $\mathbb{R}^3$ for each real number c, since the complements of these sets are open in $\mathbb{R}^3$.

Example

Let X be a subset of $\mathbb{R}^n$, and let $\mathbf{x}_0$ be a point of X. Then the sets $\{\mathbf{x} \in X : |\mathbf{x} - \mathbf{x}_0| \leq r\}$ and $\{\mathbf{x} \in X : |\mathbf{x} - \mathbf{x}_0| \geq r\}$ are closed for each non-negative real number r. In particular, the set $\{\mathbf{x}_0\}$ consisting of the single point $\mathbf{x}_0$ is a closed set in X. (These results follow immediately using Lemma 3.1 and Lemma 3.2 and the definition of closed sets.)

Let $\mathcal{A}$ be some collection of subsets of a set X. Then

$$X \setminus \bigcup_{S \in \mathcal{A}} S = \bigcap_{S \in \mathcal{A}} (X \setminus S), \qquad X \setminus \bigcap_{S \in \mathcal{A}} S = \bigcup_{S \in \mathcal{A}} (X \setminus S)$$

(i.e., the complement of the union of some collection of subsets of X is the intersection of the complements of those sets, and the complement of the intersection of some collection of subsets of X is the union of the complements of those sets).

Indeed let $\mathcal{A}$ be some collection of subsets of a set X, and let $\mathbf{x}$ be a point of X. Then

$$\begin{aligned} \mathbf{x} \in X \setminus \bigcup_{S \in \mathcal{A}} S &\iff \mathbf{x} \notin \bigcup_{S \in \mathcal{A}} S \\ &\iff \text{for all } S \in \mathcal{A},\ \mathbf{x} \notin S \\ &\iff \text{for all } S \in \mathcal{A},\ \mathbf{x} \in X \setminus S \\ &\iff \mathbf{x} \in \bigcap_{S \in \mathcal{A}} (X \setminus S), \end{aligned}$$

and therefore

$$X \setminus \bigcup_{S \in \mathcal{A}} S = \bigcap_{S \in \mathcal{A}} (X \setminus S).$$

Again let $\mathbf{x}$ be a point of X. Then

$$\begin{aligned}
\mathbf{x} \in X \setminus \bigcap_{S \in \mathcal{A}} S &\iff \mathbf{x} \notin \bigcap_{S \in \mathcal{A}} S \\
&\iff \text{there exists } S \in \mathcal{A} \text{ for which } \mathbf{x} \notin S \\
&\iff \text{there exists } S \in \mathcal{A} \text{ for which } \mathbf{x} \in X \setminus S \\
&\iff \mathbf{x} \in \bigcup_{S \in \mathcal{A}} (X \setminus S),
\end{aligned}$$

and therefore

$$X \setminus \bigcap_{S \in \mathcal{A}} S = \bigcup_{S \in \mathcal{A}} (X \setminus S).$$

The following result therefore follows directly from Proposition 3.3.

Proposition 3.6

Let X be a subset of $\mathbb{R}^n$. The collection of closed sets in X has the following properties:—

(i) *the empty set $\emptyset$ and the whole set X are both closed in X;*

(ii) *the intersection of any collection of closed sets in X is itself closed in X;*

(iii) *the union of any finite collection of closed sets in X is itself closed in X.*

Lemma 3.7

Let X be a subset of $\mathbb{R}^n$, and let F be a subset of X which is closed in X. Let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ be a sequence of points of F which converges to a point $\mathbf{p}$ of X. Then $\mathbf{p} \in F$.

Proof

The complement $X \setminus F$ of F in X is open, since F is closed. Suppose that $\mathbf{p}$ were a point belonging to $X \setminus F$. It would then follow from Lemma 3.5 that $\mathbf{x}_j \in X \setminus F$ for all values of j greater than some positive integer N, contradicting the fact that $\mathbf{x}_j \in F$ for all j. This contradiction shows that $\mathbf{p}$ must belong to F, as required. ∎

3.5. Closed Sets and Limit Points

Lemma 3.8

A subset F of n-dimensional Euclidean space $\mathbb{R}^n$ is closed in $\mathbb{R}^n$ if and only if it contains its limit points.

Proof

Let F be a closed set in $\mathbb{R}^n$ and let $\mathbf{p}$ be a limit point of F. It follows from Lemma 2.5 that there exists an infinite sequence of points of F that converges to the point $\mathbf{p}$. It then follows from Lemma 3.7 that $\mathbf{p} \in F$. Thus if the set F is closed then it contains its limit points.

3. Open and Closed Sets in Euclidean Spaces (continued)

Conversely let F be a subset of $\mathbb{R}^n$ that contains its limit points. Let $\mathbf{p} \in \mathbb{R}^n \setminus F$. Then $\mathbf{p}$ is not a limit point of F. It follows from the definition of limit points that there exists some positive real number δ for which

$$\{\mathbf{x} \in F : 0 < |\mathbf{x} - \mathbf{p}| < \delta\} = \emptyset.$$

It then follows from this that the open ball in $\mathbb{R}^n$ of radius δ about the point $\mathbf{p}$ is contained in the complement of F. We conclude therefore that the complement of F in $\mathbb{R}^n$ is open in $\mathbb{R}^n$, and thus F is closed in $\mathbb{R}^n$, as required. $\blacksquare$