MAU23203—Analysis in Several Variables School of Mathematics, Trinity College Michaelmas Term 2019 Section 5: Compactness and the Heine-Borel Theorem

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5. Compactness and the Heine-Borel Theorem

5.1. Compact Subsets of Euclidean Spaces

Definition

Let K be a subset of n-dimensional Euclidean space $\mathbb{R}^n$. A collection $\mathcal{C}$ of open sets in $\mathbb{R}^n$ is said to *cover* K if

$$K = \bigcup_{V \in \mathcal{C}} V.$$

In other words, a collection $\mathcal{C}$ of open sets in $\mathbb{R}^n$ is said to cover K if and only if each point of K belongs to at least one open set belonging to the collection $\mathcal{C}$.

Definition

A subset K of $\mathbb{R}^n$ is said to be *compact* if, given any collection of open sets in $\mathbb{R}^n$ which covers K, there exists some finite subcollection which also covers K.

Lemma 5.1

Let F and K be subsets of $\mathbb{R}^n$ where F is closed, K is compact and $F \subset K$. Then F is compact.

Proof

Let $\mathcal{C}$ be any collection of open sets in $\mathbb{R}^n$ covering F. On adjoining the open set $\mathbb{R}^n \setminus F$ to $\mathcal{C}$, we obtain a collection of open sets which covers the compact set K. The compactness of K ensures that some finite subcollection of this collection covers K. The open sets in this subcollection that belong to $\mathcal{C}$ then constitute a finite subcollection of $\mathcal{C}$ that covers F. Thus F is compact, as required. ■

Lemma 5.2

Let $\varphi\colon \mathbb{R}^m \to \mathbb{R}^n$ be a continuous function between Euclidean spaces $\mathbb{R}^m$ and $\mathbb{R}^n$, and let K be a compact subset of $\mathbb{R}^m$. Then $\varphi(K)$ is a compact subset of $\mathbb{R}^n$.

Proof

Let $\mathcal{C}$ be a collection of open sets in $\mathbb{R}^n$ which covers $\varphi(K)$. Then K is covered by the collection of all open sets of the form $\varphi^{-1}(V)$ for some $V \in \mathcal{C}$. It follows from the compactness of K that there exists a finite collection $V_1, V_2, \ldots, V_k$ of open sets belonging to $\mathcal{C}$ such that

$$K \subset \varphi^{-1}(V_1) \cup \varphi^{-1}(V_2) \cup \cdots \cup \varphi^{-1}(V_k).$$

But then $\varphi(K) \subset V_1 \cup V_2 \cup \cdots \cup V_k$. This shows that $\varphi(K)$ is compact. ∎

Lemma 5.3

Let $f \colon K \to \mathbb{R}$ be a continuous real-valued function on a compact subset K of $\mathbb{R}^n$. Then f is bounded above and below on K.

Proof

The range $f(K)$ of the function f is covered by some finite collection $I_1, I_2, \ldots, I_k$ of open intervals of the form $(-m, m)$, where $m \in \mathbb{N}$, since $f(K)$ is compact (Lemma 5.2) and $\mathbb{R}$ is covered by the collection of all intervals of this form. It follows that $f(K) \subset (-M, M)$, where $(-M, M)$ is the largest of the intervals $I_1, I_2, \ldots, I_k$. Thus the function f is bounded above and below on K, as required. ∎

Proposition 5.4

Let $f\colon K \to \mathbb{R}$ be a continuous real-valued function on a compact subset K of $\mathbb{R}^n$. Then there exist points $\mathbf{u}$ and $\mathbf{v}$ of K such that $f(\mathbf{u}) \leq f(\mathbf{x}) \leq f(\mathbf{v})$ for all $\mathbf{x} \in K$.

Proof

Let $m = \inf\{f(\mathbf{x}) : \mathbf{x} \in K\}$ and $M = \sup\{f(\mathbf{x}) : \mathbf{x} \in K\}$. There must exist $\mathbf{v} \in K$ satisfying $f(\mathbf{v}) = M$, for if $f(\mathbf{x}) < M$ for all $\mathbf{x} \in K$ then the function $\mathbf{x} \mapsto 1/(M - f(\mathbf{x}))$ would be a continuous real-valued function on K that was not bounded above, contradicting Lemma 5.3. Similarly there must exist $\mathbf{u} \in K$ satisfying $f(\mathbf{u}) = m$, since otherwise the function $\mathbf{x} \mapsto 1/(f(\mathbf{x}) - m)$ would be a continuous function on K that was not bounded above, again contradicting Lemma 5.3. But then $f(\mathbf{u}) \leq f(\mathbf{x}) \leq f(\mathbf{v})$ for all $\mathbf{x} \in K$, as required. ∎

Proposition 5.5

Let K be a compact subset of a Euclidean space $\mathbb{R}^n$. Then K is closed in $\mathbb{R}^n$.

Proof

Let $\mathbf{p}$ be a point of $\mathbb{R}^n$ that does not belong to K, and let $f(\mathbf{x}) = |\mathbf{x} - \mathbf{p}|$ for all $\mathbf{x} \in \mathbb{R}^n$. It follows from Proposition 5.4 that there is a point $\mathbf{q}$ of K such that $f(\mathbf{x}) \geq f(\mathbf{q})$ for all $\mathbf{x} \in K$, because K is compact. Now $f(\mathbf{q}) > 0$, since $\mathbf{q} \neq \mathbf{p}$. Let δ satisfy $0 < \delta \leq f(\mathbf{q})$. Then the open ball of radius δ about the point $\mathbf{p}$ is contained in the complement of K, because $f(\mathbf{x}) < f(\mathbf{q})$ for all points $\mathbf{x}$ of this open ball. It follows that the complement of K is an open set in $\mathbb{R}^n$, and thus K itself is closed in $\mathbb{R}^n$. ∎

Let F be a subset of n-dimensional Euclidean space $\mathbb{R}^n$. For each $\mathbf{x} \in \mathbb{R}^n$, we denote by $d(\mathbf{x}, F)$ the (Euclidean) distance from the point $\mathbf{x}$ to the set F. This distance $d(\mathbf{x}, F)$ is defined so that

$$d(\mathbf{x}, F) = \inf\{|\mathbf{x} - \mathbf{w}| : \mathbf{w} \in F\}.$$

Lemma 5.6

Let F be a subset of $\mathbb{R}^n$. Then

$$|d(\mathbf{x}, F) - d(\mathbf{y}, F)| \leq |\mathbf{x} - \mathbf{y}|$$

for all $\mathbf{x}, \mathbf{y} \in F$, and thus the function sending points $\mathbf{x}$ on $\mathbb{R}^n$ to their distance $d(\mathbf{x}, F)$ from the set F is a continuous real-valued function on $\mathbb{R}^n$.

Proof

Let ε be a real number satisfying $\varepsilon > 0$, and let $\mathbf{x}$ and $\mathbf{y}$ be points of $\mathbb{R}^n$. Then there exists $\mathbf{z} \in F$ for which $|\mathbf{y} - \mathbf{z}| < d(\mathbf{y}, F) + \varepsilon$. It follows from the Triangle Inequality that

$$d(\mathbf{x}, F) \leq |\mathbf{x} - \mathbf{z}| \leq |\mathbf{x} - \mathbf{y}| + |\mathbf{y} - \mathbf{z}| < |\mathbf{x} - \mathbf{y}| + d(\mathbf{y}, F) + \varepsilon$$

and thus

$$d(\mathbf{x}, F) - d(\mathbf{y}, F) < |\mathbf{x} - \mathbf{y}| + \varepsilon.$$

Now the inequality just obtained must hold for all positive real numbers ε, and the left hand side of the inequality is independent of the value of ε. It must therefore be the case that

$$d(\mathbf{x}, F) - d(\mathbf{y}, F) \leq |\mathbf{x} - \mathbf{y}|.$$

Interchanging the roles of $\mathbf{x}$ and $\mathbf{y}$, we see also that

$$d(\mathbf{y}, F) - d(\mathbf{x}, F) \leq |\mathbf{x} - \mathbf{y}|.$$

It follows that

$$|d(\mathbf{x}, F) - d(\mathbf{y}, F)| \leq |\mathbf{x} - \mathbf{y}|.$$

This inequality ensures that the function that sends points $\mathbf{x}$ of $\mathbb{R}^n$ to their distance $d(\mathbf{x}, F)$ from the set F is a continuous function on $\mathbb{R}^n$, as required. ∎

Given a subset F of $\mathbb{R}^n$ and a positive real number δ, we denote by $B(F,\delta)$ the δ-*neighbourhood* of the set F in $\mathbb{R}^n$, defined so that

$$B(F,\delta) = \{\mathbf{x} \in \mathbb{R}^n : d(\mathbf{x}, F) < \delta\}.$$

Proposition 5.7

Let K and V be subsets of $\mathbb{R}^n$, where K is compact, V is open and $K \subset V$. Then there exists some positive real number δ for which $B(K,\delta) \subset V$.

Proof

Let $F = \mathbb{R}^n \setminus V$, and let $f(\mathbf{x}) = d(\mathbf{x}, F)$ for all $\mathbf{x} \in \mathbb{R}^n$, where $d(\mathbf{x}, F)$ denotes the distance from the point $\mathbf{x}$ to the set F. Now the function f is a continuous real-valued function on $\mathbb{R}^n$. Moreover $f(\mathbf{x}) > 0$ for all $\mathbf{x} \in V$, and therefore $f(\mathbf{x}) > 0$ for all $\mathbf{x} \in K$. It then follows from Proposition 5.4 that there exists some point $\mathbf{u}$ of K with the property that $f(\mathbf{u}) \leq f(\mathbf{x})$ for all $\mathbf{x} \in K$. Let $\delta = f(\mathbf{u})$. Then $|\mathbf{x} - \mathbf{z}| \geq \delta$ for all $\mathbf{x} \in K$ and $\mathbf{z} \in F$. It follows that $B(\mathbf{x}, \delta) \subset V$ for all $\mathbf{x} \in K$, where $B(\mathbf{x}, \delta)$ denotes the open ball of radius δ centred on the point $\mathbf{x}$. Therefore $B(K, \delta) \subset V$, as required. ∎

Alternative Proof

For each point $\mathbf{w}$ of K there exists some positive real number $\delta_{\mathbf{w}}$ such that $B(\mathbf{w}, 2\delta_{\mathbf{w}}) \subset V$ where $B(\mathbf{w}, 2\delta_{\mathbf{w}})$ denotes the open ball of radius $2\delta_{\mathbf{w}}$ centred on the point $\mathbf{w}$ for each $\mathbf{w} \in K$. Now the collection $(B(\mathbf{w}, \delta_{\mathbf{w}}) : \mathbf{w} \in K)$ of open balls constitutes an open cover of the compact set K. The definition of compactness therefore ensures that there exist points $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m$ (finite in number) such that

$$K \subset \bigcup_{j=1}^{m} B(\mathbf{w}_j, \delta_{\mathbf{w}_j}).$$

Let δ be the minimum of the positive real numbers $\delta_{\mathbf{w}_j}$ for $j = 1, 2, \ldots, m$. Then $\delta > 0$. Moreover the Triangle Inequality ensures that

$$B(\mathbf{z}, \delta) \subset B(\mathbf{w}_j, 2\delta_{\mathbf{w}_j}) \subset V$$

for all $\mathbf{z} \in B(\mathbf{w}_j, \delta_{\mathbf{w}_j})$, and therefore $\bigcup_{\mathbf{z}\in K} B(\mathbf{z}, \delta) \subset V$. But $\bigcup_{\mathbf{z}\in K} B(\mathbf{z}, \delta) = B(K, \delta)$, because a point $\mathbf{x}$ of $\mathbb{R}^n$ belongs to $B(K, \delta)$ if and only if $|\mathbf{x} - \mathbf{z}| < \delta$ for some $\mathbf{z} \in K$. Thus $B(K, \delta) \subset V$, as required. $\blacksquare$

5.2. The Heine-Borel Theorem

We now show that any closed bounded interval in the real line is compact. This result is known as the *Heine-Borel Theorem*. The proof of this theorem uses the *least upper bound principle* which states that, given any non-empty set S of real numbers which is bounded above, there exists a *least upper bound* (or *supremum*) $\sup S$ for the set S.

Theorem 5.8

(Heine-Borel in One Dimension) Let a and b be real numbers satisfying $a < b$. Then the closed bounded interval $[a, b]$ is a compact subset of $\mathbb{R}$.

Proof

Let $\mathcal{C}$ be a collection of open sets in $\mathbb{R}$ with the property that each point of the interval $[a, b]$ belongs to at least one of these open sets. We must show that $[a, b]$ is covered by finitely many of these open sets.

Let S be the set of all $\tau \in [a, b]$ with the property that $[a, \tau]$ is covered by some finite collection of open sets belonging to $\mathcal{C}$, and let $s = \sup S$. Now $s \in W$ for some open set W belonging to $\mathcal{C}$. Moreover W is open in $\mathbb{R}$, and therefore there exists some $\delta > 0$ such that $(s - \delta, s + \delta) \subset W$. Moreover $s - \delta$ is not an upper bound for the set S, hence there exists some $\tau \in S$ satisfying $\tau > s - \delta$. It follows from the definition of S that $[a, \tau]$ is covered by some finite collection $V_1, V_2, \ldots, V_r$ of open sets belonging to $\mathcal{C}$.

Let $t \in [a, b]$ satisfy $\tau \leq t < s + \delta$. Then

$$[a, t] \subset [a, \tau] \cup (s - \delta, s + \delta) \subset V_1 \cup V_2 \cup \cdots \cup V_r \cup W,$$

and thus $t \in S$. In particular $s \in S$, and moreover $s = b$, since otherwise s would not be an upper bound of the set S. Thus $b \in S$, and therefore $[a, b]$ is covered by a finite collection of open sets belonging to $\mathcal{C}$, as required. ∎

Definition

We define a *closed n-dimensional block* in $\mathbb{R}^n$ to be a subset of $\mathbb{R}^n$ that is a product of closed bounded intervals.

Thus a subset K of $\mathbb{R}^n$ is a closed n-dimensional block if and only if there exist real numbers $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ such that $a_i \leq b_i$ for $i = 1, 2, \ldots, n$ and

$$K = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n].$$

Proposition 5.9

A closed n-dimensional block is a compact set.

Proof

We prove the result by induction on the dimension n of the Euclidean space. The result when $n = 1$ is the one-dimensional Heine-Borel Theorem (Theorem 5.8). Thus suppose as our induction hypothesis that $n > 1$ and that that every closed $(n-1)$-dimensional block in $\mathbb{R}^{n-1}$ is a compact set. Let K be an n-dimensional block in $\mathbb{R}^n$, and let

$$K = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n],$$

where $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ are real numbers that satisfy $a_i \leq b_i$ for $i = 1, 2, \ldots, n$.

Let $p\colon \mathbb{R}^n \to \mathbb{R}$ be the projection function defined such that

$$p(x_1, x_2, \ldots, x_n) = x_n$$

for all $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. The induction hypothesis then ensures that K_z is a compact set for all $z \in [a_n, b_n]$, where

$$K_z = \{\mathbf{x} \in K : p(\mathbf{x}) = z\}.$$

Let $\mathcal{C}$ be a collection of open sets in $\mathbb{R}^n$ that covers K. The compactness of K_z ensures that, for each real number z satisfying $a_n \leq z \leq b_n$ there exists a finite subcollection $\mathcal{C}_z$ of $\mathcal{C}$ such that $K_z \subset \bigcup_{V \in \mathcal{C}_z} V$. Let $W_z = \bigcup_{V \in \mathcal{C}_z} V$. (The set W_z is thus the union of the open sets belonging to the finite subcollection $\mathcal{C}_z$ of $\mathcal{C}$.)

Now K_z is compact, W_z is open, and $K_z \subset W_z$. It follows that there exists some positive real number δ_z such that $B(K, \delta_z) \subset W_z$, where $B(K, \delta_z)$ denotes the δ-neighbourhood of K in $\mathbb{R}^n$ i.e., the subset of $\mathbb{R}^n$ consisting of those points of $\mathbb{R}^n$ that lie within a distance δ_z of the set K_z (see Proposition 5.7). But then

$$\{\mathbf{x} \in K : z - \delta_z < p(\mathbf{x}) < z + \delta_z\} \subset W_z$$

for all $z \in [a_n, b_n]$. Now the collection of all open intervals in $\mathbb{R}$ that are of the form $(z - \delta_z, z + \delta_z)$ constitute an open cover of the closed bounded interval $[a_n, b_n]$. It follows from the one-dimensional Heine-Borel Theorem (Theorem 5.8) that there exist $z_1, z_2, \ldots, z_m \in [a_n, b_n]$ such that

$$[a_n, b_n] \subset \bigcup_{j=1}^{m} (z_j - \delta_{z_j}, z_j + \delta_{z_j}).$$

But then

$$K \subset \bigcup_{j=1}^{n} W_{z_j}.$$

Moreover $\bigcup_{j=1}^{n} W_{z_j}$ is the union of all the open sets that belong to the collection $\mathcal{D}$ obtained by amalgamating the finite collections $\mathcal{C}_{z_1}, \mathcal{C}_{z_2}, \ldots, \mathcal{C}_{z_m}$. Then $\mathcal{D}$ is a finite subcollection of $\mathcal{C}$ which covers the n-dimensional block K. The result follows. ∎

Theorem 5.10 (Multidimensional Heine-Borel Theorem)

A subset of a Euclidean space is compact if and only if it is both closed and bounded.

Proof

Let K be a compact subset of n-dimensional Euclidean space. The function that maps each point $\mathbf{x}$ of $\mathbb{R}^n$ to its Euclidean distance $|\mathbf{x}|$ from the origin is then a bounded function on K (Lemma 5.3) and therefore K is a bounded set. Moreover it follows from Proposition 5.5 that K is closed in $\mathbb{R}^n$.

Conversely let K be a subset of $\mathbb{R}^n$ that is both closed and bounded. Then there exists some positive real number R large enough to ensure that $K \subset H$, where

$$H = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : -R \leq x_i \leq R \text{ for } i = 1, 2, \ldots, n\}.$$

Now H is a closed n-dimensional block in $\mathbb{R}^n$. It follows from Proposition 5.9 that H is a compact subset of $\mathbb{R}^n$. Thus K is a closed subset of a compact set. It follows from Lemma 5.1 that K is a compact subset of $\mathbb{R}^n$, as required. ∎