MAU22200—Advanced Analyis
School of Mathematics, Trinity College
Hilary Term 2020
Section 10: Stieltjes Measure and Integration

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10.1. Stieltjes Content

Lemma 10.1

Let $F\colon \mathbb{R} \to \mathbb{R}$ be a non-decreasing function of a real variable. Then, for each real number s, there are well-defined real numbers $F(s^-)$ and $F(s^+)$ characterized by the properties that

$$F(s^-) = \lim_{x \to s^-} F(x) = \sup\{F(x) : x < s\}$$

and

$$F(s^+) = \lim_{x \to s^+} F(x) = \inf\{F(x) : x > s\}.$$

Moreover $F(s^-) \leq F(s) \leq F(s^+)$ for all real numbers s, and $F(u^+) \leq F(v) \leq F(w^-)$ for all real numbers u, v and w satisfying $u < v < w$.

Proof

Let s be a real number. The set $\{F(x) : x < s\}$ is non-empty, and is bounded above by $F(s)$. This set therefore has a least upper bound $F(s^-)$, and moreover $F(s^-) \leq F(s)$.

Now let ε be any strictly positive real number. Then $F(s^-) - \varepsilon$ is not an upper bound for the set $\{F(x) : x < s\}$, because $F(s^-)$ is the least upper bound of this set. It follows that there exists some strictly positive real number δ for which $F(s - \delta) > F(s^-) - \varepsilon$. Then $F(s^-) - \varepsilon < F(x) \leq F(s^-)$ for all real numbers x satisfying $s - \delta < x < s$. It follows that $F(s^-) = \lim_{x \to s^-} F(x)$. An analogous argument shows that the set $\{F(x) : x > s\}$ has a greatest lower bound $F(s^+)$, and moreover $F(s^+) \geq F(s)$ and $F(s^+) = \lim_{x \to s^+} F(x)$.

Now let u, v and w be real numbers satisfying $u < v < w$, Then the real numbers $F(u^+)$ and $F(w^-)$ are the greatest lower bound and least upper bound of the sets $\{F(x) : x > u\}$ and $\{F(x) : x < w\}$, respectively, and $F(v)$ belongs to both of these sets. It follows that $F(u^+) \leq F(v) \leq F(w^-)$, as required. ∎

The definition of $F(s^+)$ and $F(s^-)$ for each real number s ensures that, given any given any real number s and any strictly positive real number ε, there exist real numbers q and r satisfying $q < s < r$ for which $F(q) > F(s^-) - \varepsilon$ and $F(r) < F(s^+) + \varepsilon$.

Definition

Let $F\colon \mathbb{R} \to \mathbb{R}$ be a non-decreasing function of a real variable. The *Stieltjes content* $m_F(I)$ of each bounded interval or singleton set I contained in $\mathbb{R}$ with respect to the function F is then defined so that

$$\begin{aligned} m_F(\{v\}) &= F(v^+) - F(v^-), \\ m_F([u,v]) &= F(v^+) - F(u^-), \\ m_F([u,v)) &= F(v^-) - F(u^-), \\ m_F((u,v]) &= F(v^+) - F(u^+), \\ m_F((u,v)) &= F(v^-) - F(u^+) \end{aligned}$$

for all real numbers u and v satisfying $u < v$.

Proposition 10.2

Let $F\colon \mathbb{R} \to \mathbb{R}$ be a non-decreasing function of a real-variable and, for any singleton set or bounded interval J, let $m_F(J)$ denote the Stieltjes content of J with respect to the function F. Let a and b be real numbers satisfying $a < b$, and let $u_0, u_1, \ldots, u_N$ be a list of real numbers with the property that

$$a = u_0 < u_1 < u_2 < \cdots < u_N = b.$$

For each integer j between 0 and N, let $D_j = \{u_j\}$, and, for each integer j between 1 and N, let

$$E_j = (u_{j-1}, u_j) = \{x \in \mathbb{R} : u_{j-1} < x < u_j\}.$$

Then

$$m_F((a,b)) = \sum_{j=1}^{N-1} m_F(D_j) + \sum_{j=1}^{N} m_F(E_j).$$

Also

$$\begin{aligned} m_F([a,b)) &= \sum_{j=0}^{N-1} m_F(D_j) + \sum_{j=1}^{N} m_F(E_j), \\ m_F((a,b]) &= \sum_{j=1}^{N} m_F(D_j) + \sum_{j=1}^{N} m_F(E_j), \\ m_F([a,b]) &= \sum_{j=0}^{N} m_F(D_j) + \sum_{j=1}^{N} m_F(E_j). \end{aligned}$$

Proof

$$\begin{aligned} m_F((a,b)) &= F(b^-) - F(a^+) = F(u_N^-) - F(u_0^+) \\ &= F(u_{N-1}^+) - F(u_0^+) + F(u_N^-) - F(u_{N-1}^+) \\ &= \sum_{j=1}^{N-1} (F(u_j^+) - F(u_{j-1}^+)) + m_F(E_N) \\ &= \sum_{j=1}^{N-1} (m_F(D_j) + m_F(E_j)) + m_F(E_N) \\ &= \sum_{j=1}^{N-1} m_F(D_j) + \sum_{j=1}^{N} m_F(E_j). \end{aligned}$$

Then

$$
\begin{aligned}
m_F([a,b)) &= F(b^-) - F(a^-) \\
&= F(a^+) - F(a^-) + F(b^-) - F(a^-) \\
&= m_F(D_0) + m_F((a,b)) \\
&= \sum_{j=0}^{N-1} m_F(D_j) + \sum_{j=1}^{N} m_F(E_j),
\end{aligned}
$$

$$
\begin{aligned}
m_F((a,b]) &= F(b^+) - F(a^+) \\
&= F(b^+) - F(b^-) + F(b^-) - F(a^+) \\
&= m_F(D_N) + m_F((a,b)) \\
&= \sum_{j=1}^{N} m_F(D_j) + \sum_{j=1}^{N} m_F(E_j)
\end{aligned}
$$

and

$$\begin{aligned} m_F([a,b]) &= F(b^+) - F(a^-) \\ &= F(a^+) - F(a^-) + F(b^+) - F(a^+) \\ &= m_F(D_0) + m_F((a,b]) \\ &= \sum_{j=0}^{N} m_F(D_j) + \sum_{j=1}^{N} m_F(E_j), \end{aligned}$$

This establishes all the required identities. ∎

Proposition 10.3

Let $F\colon \mathbb{R} \to \mathbb{R}$ be a non-decreasing function of a real-variable and, for any singleton set or bounded interval J, let $m_F(J)$ denote the Stieltjes content of J with respect to the function F. Let a and b be real numbers satisfying $a < b$, and let $u_0, u_1, \ldots, u_N$ be a list of real numbers with the property that

$$a = u_0 < u_1 < u_2 < \cdots < u_N = b.$$

For each integer j between 0 and N, let $D_j = \{u_j\}$, and, for each integer j between 1 and N, let

$$E_j = (u_{j-1}, u_j) = \{x \in \mathbb{R} : u_{j-1} < x < u_j\}.$$

Also let J be an interval or singleton set whose endpoints are included in the list $u_0, u_1, \ldots, u_N$, and let

$$\begin{aligned} S(J) &= \{j \in \mathbb{Z} : 0 \leq j \leq N \text{ and } D_j \subset J\}, \\ T(J) &= \{j \in \mathbb{Z} : 1 \leq j \leq N \text{ and } E_j \subset J\}. \end{aligned}$$

Then

$$m_F(J) = \sum_{j \in S(J)} m_F(D_j) + \sum_{j \in T(J)} m_F(E_j).$$

Proof

An integer j between 0 and N belongs to $S(J)$ if and only if $u_j \in J$, and an integer j between 1 and N belongs to $T(J)$ if and only if $(u_{j-1}, u_j) \subset J$.

The proof is accomplished through a case-by-case analysis.

First suppose that J is a singleton set. Then $J = \{u_k\}$ for some integer k between 1 and N. In this case $S(J) = \{u_k\}$, $T(J) = \emptyset$ and

$$m_F(J) = m_F = m_F(D_k) = \sum_{j \in S(J)} m_F(D_j) + \sum_{j \in T(J)} m_F(E_j).$$

In the remaining cases, suppose that J takes one of the forms (a, b), $[a, b)$, $(a, b]$ or $[a, b]$, where a and b are real numbers and $a < b$. There then exist integers p and q between 1 and N satisfying $p < q$ for which $a = u_p$ and $b = u_q$.
Suppose then that $J = (a, b) = (u_p, u_q)$. Then

$$S(J) = \{k \in \mathbb{Z} : p < k < q\} \quad \text{and} \quad T(J) = \{k \in \mathbb{Z} : p < k \leq q\}.$$

Then Proposition 10.2 ensures that

$$\begin{aligned} m_F((a, b)) &= \sum_{j=p+1}^{q-1} m_F(D_j) + \sum_{j=p+1}^{q} m_F(E_j) \\ &= \sum_{j \in S(J)} m_F(D_j) + \sum_{j \in T(J)} m_F(E_j). \end{aligned}$$

The same strategy applies in the remaining cases. In the case where $J = [a, b) = [u_p, u_q)$ we have

$$S(J) = \{k \in \mathbb{Z} : p \leq k < q\} \quad \text{and} \quad T(J) = \{k \in \mathbb{Z} : p < k \leq q\},$$

in the case where $J = (a, b] = (u_p, u_q]$ we have

$$S(J) = \{k \in \mathbb{Z} : p < k \leq q\} \quad \text{and} \quad T(J) = \{k \in \mathbb{Z} : p < k \leq q\},$$

in the case where $J = [a, b] = [u_p, u_q]$ we have

$$S(J) = \{k \in \mathbb{Z} : p \leq k \leq q\} \quad \text{and} \quad T(J) = \{k \in \mathbb{Z} : p < k \leq q\},$$

and in each of these three cases the required identity follows on applying the relevant identity stated in Proposition 10.2. ∎

Proposition 10.4

Let $F\colon \mathbb{R} \to \mathbb{R}$ be a non-decreasing function of a real-variable and, for any singleton set or bounded interval K, let $m_F(K)$ denote the Stieltjes content of K with respect to the function F. Also J, $J^{(1)}, J^{(2)}, \ldots, J^{(s)}$ be bounded intervals or singleton sets contained in the set $\mathbb{R}$ of real numbers. Suppose that $J^{(1)}, J^{(2)}, \ldots, J^{(s)}$ are pairwise disjoint and that $J = \bigcup_{r=1}^{s} J^{(r)}$. Then

$$m_F(J) = \sum_{r=1}^{s} m_F(J^{(r)}).$$

Proof

Let $u_0, u_1, \ldots, u_N$ be a list of real numbers, listed in increasing order, that contains the endpoints of each of the singleton sets or bounded intervals J, $J^{(1)}, J^{(2)}, \ldots, J^{(s)}$. For each integer j between 0 and N, let $D_j = \{u_j\}$, and, for each integer j between 1 and N, let

$$E_j = (u_{j-1}, u_j) = \{x \in \mathbb{R} : u_{j-1} < x < u_j\}.$$

Also, for each interval or singleton set K whose endpoints are included in the list $u_0, u_1, \ldots, u_N$, let

$$\begin{aligned} S(K) &= \{j \in \mathbb{Z} : 0 \leq j \leq N \text{ and } D_j \subset I\}, \\ T(K) &= \{j \in \mathbb{Z} : 1 \leq j \leq N \text{ and } E_j \subset I\}. \end{aligned}$$

Then

$$m_F(K) = \sum_{j \in S(K)} m_F(D_j) + \sum_{j \in T(K)} m_F(E_j)$$

for any such interval K.

In particular

$$m_F(J) = \sum_{j \in S(J)} m_F(D_j) + \sum_{j \in T(J)} m_F(E_j)$$

and

$$m_F(J^{(r)}) = \sum_{j \in S(J^{(r)})} m_F(D_j) + \sum_{j \in T(J^{(r)})} m_F(E_j)$$

for $r = 1, 2, \ldots, s$.

Now the sets $J^{(1)}, J^{(2)}, \ldots, J^{(s)}$ are pairwise disjoint, and the union of these pairwise disjoint sets is the set J. It follows that if j is an integer between 0 and N for which $D_j \subset J$ then there is exactly one integer r between 1 and s for which $D_j \subset J^{(r)}$, and therefore each integer j in $S(J)$ belongs to exactly one of the sets $S(J^{(1)}), S(J^{(2)}), \ldots, S(J^{(s)})$. Similarly if j is an integer between 1 and N for which $E_j \subset J$ then there is exactly one integer r between 1 and s for which $E_j \subset J^{(r)}$. and therefore each integer j in $T(J)$ belongs to exactly one of the sets $T(J^{(1)}), T(J^{(2)}), \ldots, T(J^{(s)})$. It follows that

$$
\begin{aligned}
m_F(J) &= \sum_{j \in S(J)} m_F(D_j) + \sum_{j \in T(J)} m_F(E_j) \\
&= \sum_{r=1}^{s} \sum_{j \in S(J^{(r)})} m_F(D_j) + \sum_{r=1}^{s} \sum_{j \in T(J^{(r)})} m_F(E_j) \\
&= \sum_{r=1}^{s} m_F(J^{(r)}),
\end{aligned}
$$

as required. ∎

The following two propositions are the analogues, for Stieltjes measures, of Proposition 7.5 and Proposition 7.6.

Proposition 10.5

Let $F\colon \mathbb{R} \to \mathbb{R}$ be a non-decreasing function of a real-variable and, for any singleton set or bounded interval K, let $m_F(K)$ denote the Stieltjes content of K with respect to the function F. Let J be a bounded interval or singleton set in the real line $\mathbb{R}$, and let $J_1, J_2, \ldots, J_s$ be a finite collection of sets each of which is a bounded interval or singleton set in $\mathbb{R}$. Suppose that $J \subset \bigcup_{k=1}^{s} J_k$.

Then $m_F(J) \leq \sum_{k=1}^{s} m_F(J_k)$.

Proof

The collection of subsets of $\mathbb{R}$ consisting of the empty set, the singleton sets that are of the form $\{c\}$ for some real number c, and the bounded intervals is a semiring of subsets of $\mathbb{R}$. Proposition 10.4 establishes that Stieljes content is finitely additive on this semiring and is thus a true content function on the semiring. The required result therefore follows immediately on applying Proposition 6.19. ∎

Proposition 10.6

Let $F\colon \mathbb{R} \to \mathbb{R}$ be a non-decreasing function of a real-variable and, for any singleton set or bounded interval K, let $m_F(K)$ denote the Stieltjes content of K with respect to the function F. Let J be a bounded interval or singleton set in the real line $\mathbb{R}$, and let $J_1, J_2, \ldots, J_s$ be a finite collection of sets each of which is a bounded interval or singleton set in $\mathbb{R}$. Suppose that the sets $J_1, J_2, \ldots, J_s$ are pairwise disjoint and are contained in J. Then $\sum_{k=1}^{s} m_F(J_k) \leq m_F(J)$.

Proof

The collection of subsets of $\mathbb{R}$ consisting of the empty set, the singleton sets that are of the form $\{c\}$ for some real number c, and the bounded intervals is a semiring of subsets of $\mathbb{R}$. Proposition 10.4 establishes that Stieljes content is finitely additive on this semiring and is thus a true content function on the semiring. The required result therefore follows immediately on applying Proposition 6.20. ∎

Lemma 10.7

Let $F\colon \mathbb{R} \to \mathbb{R}$ be a non-decreasing function of a real-variable. Let $\{v\}$ be a singleton set in the real line. Then, given any positive real number ε, there exists an open interval V such that $v \in V$ and $m_F(V) < m_F(\{v\}) + \varepsilon$, where $m_F(\{v\})$ and $m_F(V)$ denote the Stieltjes content of the sets $\{v\}$ and V respectively with respect to the function F.

Proof

The Stieltjes measure $m_F(\{v\})$ of the singleton set $\{v\}$ is defined by the identity $m_F(\{v\}) = F(v^+) - F(v^-)$, where

$$F(v^+) = \inf\{F(x) : x > v\} \quad \text{and} \quad F(v^-) = \inf\{F(x) : x < v\}$$

(see Lemma 10.1). It follows that, given any strictly positive real number ε, there exist real numbers u and w satisfying $u < v < w$ for which $F(u) > F(v^-) - \frac{1}{2}\varepsilon$ and $F(w) < F(v^+) + \frac{1}{2}\varepsilon$. Let $V = (u, w)$. Then V is an open interval and

$$\begin{aligned} m_F(V) &= F(w^-) - F(u^+) \leq F(w) - F(u) \\ &< F(v^+) - F(v^-) + \varepsilon = m_F(\{v\}) + \varepsilon, \end{aligned}$$

as required. ∎

Lemma 10.8

Let $F\colon \mathbb{R} \to \mathbb{R}$ be a non-decreasing function of a real-variable. Let J be a bounded interval of positive length in the real line, and let $a = \inf J$ and $b = \sup J$. Then, given any positive real number ε, there exists an open interval V such that $J \subset V$ and $m_F(V) < m_F(J) + \varepsilon$, where $m_F(J)$ and $m_F(V)$ denote the Stieltjes content of the sets J and V respectively with respect to the function F.

Proof

The endpoints a and b of the interval J satisfy $a < b$, and J coincides with exactly one of the intervals (a, b), $[a, b)$, $(a, b]$ and $[a, b]$. And the Stieltjes measures of these intervals are defined so that

$$m_F((a, b) = F(b^-) - F(a^+), \quad m_F([a, b) = F(b^-) - F(a^-),$$

$$m_F((a, b] = F(b^+) - F(a^+), \quad m_F([a, b] = F(b^+) - F(a^-).$$

Also the definitions of $F(a^-)$ and $F(b^+)$ ensure that there exist real numbers u and v satisfying $u < a < b < v$ for which $F(u) > F(a^-) - \frac{1}{2}\varepsilon$ and $F(w) < F(b^+) + \frac{1}{2}\varepsilon$.

In the case where $J = (a, b)$ we can take $V = J$.

Suppose next that $J = [a, b)$. In this case take $V = (u, b)$. Then $m_F(J) = F(b^-) - F(a^-)$ and

$$\begin{aligned} m_F(V) &= F(b^-) - F(u^+) \leq F(b^-) - F(u) \\ &< F(b^-) - F(a^-) + \tfrac{1}{2}\varepsilon < m_F(J) + \varepsilon. \end{aligned}$$

Next suppose next that $J = (a, b]$. In this case take $V = (a, w)$. Then $m_F(J) = F(b^+) - F(a^+)$ and

$$\begin{aligned} m_F(V) &= F(w^-) - F(a^+) \leq F(w) - F(a^+) \\ &< F(b^+) - F(a^+) + \tfrac{1}{2}\varepsilon < m_F(J) + \varepsilon. \end{aligned}$$

Finally suppose next that $J = [a, b]$. In this case take $V = (u, w)$. Then $m_F(J) = F(b^+) - F(a^-)$ and

$$\begin{aligned} m_F(V) &= F(w^-) - F(u^+) \leq F(w) - F(u) \\ &< F(b^+) - F(a^-) + \varepsilon = m_F(J) + \varepsilon. \end{aligned}$$

We have now verified the existence of the open set V with the required properties in all cases. $\blacksquare$

Lemma 10.9

Let $F\colon \mathbb{R} \to \mathbb{R}$ be a non-decreasing function of a real-variable. Let J be a bounded interval or singleton set in the real line, and let $a = \inf J$ and $b = \sup J$. Then, given any positive real number ε, there exists a closed interval K such that $m_F(K) > m_F(J) + \varepsilon$, where $m_F(J)$ and $m_F(K)$ denote the Stieltjes content of the sets J and K respectively with respect to the function F.

Proof

The endpoints a and b of the interval J satisfy $a \leq b$, and either J is a singleton set or else J coincides with exactly one of the intervals (a, b), $[a, b)$, $(a, b]$ and $[a, b]$. And the Stieltjes measures of these intervals are defined so that

$$m_F((a, b) = F(b^-) - F(a^+), \quad m_F([a, b) = F(b^-) - F(a^-),$$

$$m_F((a, b] = F(b^+) - F(a^+), \quad m_F([a, b] = F(b^+) - F(a^-).$$

Also the definitions of $F(a^+)$ and $F(b^-)$ ensure that there exist real numbers u and v satisfying $a < u < v < b$ for which $F(u) < F(a^+) + \frac{1}{2}\varepsilon$ and $F(w) > F(b^-) - \frac{1}{2}\varepsilon$.

In the case where J is a singleton set or a closed interval we can take $K = J$.

Suppose next that $J = (a, b]$. In this case take $K = [u, b]$. Then $m_F(J) = F(b^+) - F(a^+)$ and

$$\begin{aligned} m_F(K) &= F(b^+) - F(u^-) \geq F(b^+) - F(u) \\ &> F(b^+) - F(a^+) - \tfrac{1}{2}\varepsilon > m_F(J) - \varepsilon. \end{aligned}$$

Next suppose next that $J = [a, b)$. In this case take $K = [a, w]$. Then $m_F(J) = F(b^-) - F(a^-)$ and

$$\begin{aligned} m_F(K) &= F(w^+) - F(a^-) \geq F(w) - F(a^-) \\ &> F(b^-) - F(a^-) - \tfrac{1}{2}\varepsilon > m_F(J) - \varepsilon. \end{aligned}$$

10. Stieltjes Measure (continued)

Finally suppose next that $J = (a, b)$. In this case take $V = [u, w]$. Then $m_F(J) = F(b^-) - F(a^+)$ and

$$\begin{aligned} m_F(K) &= F(w^+) - F(u^-) \geq F(w) - F(u) \\ &> F(b^-) - F(a^+) - \varepsilon = m_F(J) - \varepsilon. \end{aligned}$$

We have now verified the existence of the open set V with the required properties in all cases. ∎

Proposition 10.10

Let $F\colon \mathbb{R} \to \mathbb{R}$ be a non-decreasing function of a real-variable and, for any singleton set or bounded interval K, let $m_F(K)$ denote the Stieltjes content of K with respect to the function F. Let I be a bounded interval or singleton set in the real line $\mathbb{R}$, and let $\mathcal{C}$ be a countable collection of subsets of $\mathbb{R}$ each of which is a bounded interval or singleton set. Suppose that $I \subset \bigcup_{B \in \mathcal{C}} J$. Then $m_F(I) \leq \sum\limits_{B \in \mathcal{C}} m_F(J)$.

Proof

There is nothing to prove if $\sum_{J \in \mathcal{C}} m(B) = +\infty$. We may therefore restrict our attention to the case where $\sum_{J \in \mathcal{C}} m(B) < +\infty$.

Moreover the result is an immediate consequence of Proposition 10.5 if the collection $\mathcal{C}$ is finite. It therefore only remains to prove the result in the case where the collection $\mathcal{C}$ is infinite, but countable.

In that case there exists an infinite sequence $J_1, J_2, J_3, \ldots$ of sets, each of which is a bounded interval or singleton set, with the property that each set in the collection $\mathcal{C}$ occurs exactly once in the sequence. Let some positive real number ε be given. It follows from Lemma 10.9 that there exists a closed interval or singleton set K such that $K \subset I$ and $m_F(K) \geq m_F(I) - \varepsilon$. Also, for each $k \in \mathbb{N}$, it follows from Lemma 10.7 and Lemma 10.8 that there exists a bounded open interval V_k such that $J_k \subset V_k$ and $m_F(V_k) < m_F(J_k) + 2^{-k}\varepsilon$. Then $K \subset \bigcup_{k=1}^{+\infty} V_k$, and thus $\{V_1, V_2, V_3, \ldots\}$ is a collection of open sets in the real line $\mathbb{R}$ which covers the closed bounded set K. It follows from the compactness of K that there exists a finite collection $k_1, k_2, \ldots, k_s$ of positive integers such that $K \subset V_{k_1} \cup V_{k_2} \cup \cdots \cup V_{k_s}$. It then follows from Proposition 10.5 that

$$m_F(K) \leq m_F(V_{k_1}) + m_F(V_{k_2}) + \cdots + m_F(V_{k_s}).$$

Now

$$\frac{1}{2^{k_1}} + \frac{1}{2^{k_2}} + \cdots + \frac{1}{2^{k_s}} \leq \sum_{k=1}^{+\infty} \frac{1}{2^k} = 1,$$

and therefore

$$\begin{aligned} m_F(K) &\leq m_F(V_{k_1}) + m_F(V_{k_2}) + \cdots + m_F(V_{k_s}) \\ &\leq m_F(J_{k_1}) + m_F(J_{k_2}) + \cdots + m_F(J_{k_s}) + \varepsilon \\ &\leq \sum_{k=1}^{+\infty} m_F(J_k) + \varepsilon. \end{aligned}$$

Also $m_F(A) < m_F(K) + \varepsilon$. It follows that

$$m_F(I) \leq \sum_{k=1}^{+\infty} m_F(J_k) + 2\varepsilon.$$

Moreover this inequality holds no matter how small the value of the positive real number ε. It follows that

$$m_F(I) \leq \sum_{k=1}^{+\infty} m_F(J_k),$$

as required. ∎

10.2. Lebesgue-Stieltjes Outer Measure

Let $\mathcal{J}$ be the semiring of subsets of the real line consisting of the empty set together with all singleton sets and bounded intervals contained in the set $\mathbb{R}$ of real numbers. Also let the empty set be assigned Stieltjes content equal to zero, so that $m_F(\emptyset) = 0$. Then Stieltjes measure determines a finitely additive content function $m_F \colon \mathcal{J} \to [0, +\infty)$ on the semiring $\mathcal{J}$ (see Proposition 10.4). The result of (f) Moreover this content function is countably subadditive. (Proposition 10.10).

We say that a collection $\mathcal{C}$ of subsets of the real line $\mathbb{R}$ *covers* a subset E of $\mathbb{R}$ if $E \subset \bigcup_{J \in \mathcal{C}} J$, (where $\bigcup_{J \in \mathcal{C}} J$ denotes the union of all the sets belonging to the collection $\mathcal{C}$). Given any subset E of $\mathbb{R}$, we shall denote by $\mathbf{CCI}(E)$ the set of all countable collections, made up of bounded intervals and singleton sets, that cover the set E.

Definition

Let $F\colon \mathbb{R} \to \mathbb{R}$ be a non-decreasing function of a real-variable and, for any singleton set or bounded interval K, let $m_F(K)$ denote the Stieltjes content of K with respect to the function F. Let E be a subset of $\mathbb{R}$. We define the *Lebesgue-Stieltjes outer measure* $\mu_F^*(E)$ of E to be the infimum, or greatest lower bound, of the quantities $\sum_{J \in \mathcal{C}} m_F(J)$, where this infimum is taken over all countable collections $\mathcal{C}$, made up of bounded intervals and singleton sets, that cover the set E. Thus

$$\mu_F^*(E) = \inf \left\{ \sum_{J \in \mathcal{C}} m_F(J) : \mathcal{C} \in \mathbf{CCI}(E) \right\}.$$

Let $F\colon \mathbb{R} \to \mathbb{R}$ be a non-decreasing function of a real-variable and, for any singleton set or bounded interval J, let $m_F(J)$ denote the Stieltjes content of J with respect to the function F. The Lebesgue-Stieltjes outer measure $\mu_F^*(E)$ of a subset E of the real line $\mathbb{R}$ is then the greatest extended real number l with the property that $l \leq \sum_{J \in \mathcal{C}} m_F(J)$ for any countable collection $\mathcal{C}$, made up of bounded intervals and singleton sets, that covers the set E. In particular, $\mu_F^*(E) = +\infty$ if and only if $\sum_{J \in \mathcal{C}} m_F(J) = +\infty$ for every countable collection $\mathcal{C}$, made up of bounded intervals and singleton sets, that covers the set E.

Note that $\mu_F^*(E) \geq 0$ for all subsets E of $\mathbb{R}$.

Lemma 10.11

Let $F\colon \mathbb{R} \to \mathbb{R}$ be a non-decreasing function of a real-variable and, for any singleton set or bounded interval K, let $m_F(K)$ denote the Stieltjes content of K with respect to the function F. Let E be a bounded interval or singleton set in $\mathbb{R}$. Then $\mu_F^*(E) = m_F(E)$, where $m_F(E)$ is the content of the set E.

Proof

It follows from Proposition 10.10 that $m_F(E) \leq \sum_{J \in \mathcal{C}} m_F(J)$ for any countable collection, made up of bounded intervals and singleton sets, that covers the set E. Therefore $m_F(E) \leq \mu_F^*(E)$. But the collection $\{E\}$ made up of the single set E is itself a countable collection of bounded intervals or singleton sets covering E, and therefore $\mu_F^*(E) \leq m_F(E)$. It follows that $\mu_F^*(E) = m_F(E)$, as required. ∎

Lemma 10.12

Let $F\colon \mathbb{R} \to \mathbb{R}$ be a non-decreasing function of a real-variable and, for any singleton set or bounded interval K, let $m_F(K)$ denote the Stieltjes content of K with respect to the function F. Let E and G be subsets of $\mathbb{R}$. Suppose that $E \subset F$. Then $\mu_F^*(E) \leq \mu_F^*(G)$.

Proof

Any countable collection, made up of bounded intervals and singleton sets, that covers the set G will also cover the set E, and therefore $\mathbf{CCI}(G) \subset \mathbf{CCI}(E)$. It follows that

$$\begin{aligned} \mu_F^*(G) &= \inf\left\{ \sum_{J \in \mathcal{C}} m_F(J) : \mathcal{C} \in \mathbf{CCI}(G) \right\} \\ &\geq \inf\left\{ \sum_{J \in \mathcal{C}} m_F(J) : \mathcal{C} \in \mathbf{CCI}(E) \right\} = \mu_F^*(E), \end{aligned}$$

as required. ∎

Proposition 10.13

Let $F\colon \mathbb{R} \to \mathbb{R}$ be a non-decreasing function of a real-variable and, for any singleton set or bounded interval K, let $m_F(K)$ denote the Stieltjes content of K with respect to the function F. Let $\mathcal{E}$ be a countable collection of subsets of $\mathbb{R}$. Then

$$\mu_F^*\left(\bigcup_{E\in\mathcal{E}} E\right) \leq \sum_{E\in\mathcal{E}} \mu_F^*(E).$$

Proof

Let $K = \mathbb{N}$ in the case where the countable collection $\mathcal{E}$ is infinite, and let $K = \{1, 2, \ldots, m\}$ in the case where the collection $\mathcal{E}$ is finite and has m elements. Then there exists a bijective function $\varphi\colon K \to \mathcal{E}$. We define $E_k = \varphi(k)$ for all $k \in K$. Then $\mathcal{E} = \{E_k : k \in K\}$, and any subset of $\mathbb{R}$ belonging to the collection $\mathcal{E}$ is of the form E_k for exactly one element k of the indexing set K.

Let some positive real number ε be given. Then corresponding to each element k of K there exists a countable collection $\mathcal{C}_k$, made up of bounded intervals and singleton sets, covering the set E_k for which

$$\sum_{J \in \mathcal{C}_k} m_F(J) < \mu_F^*(E_k) + \frac{\varepsilon}{2^k}.$$

Let $\mathcal{C} = \bigcup_{k \in K} \mathcal{C}_k$. Then $\mathcal{C}$ is a collection, made up of bounded intervals and singleton sets, that covers the union $\bigcup_{E \in \mathcal{E}} E$ of all the sets in the collection $\mathcal{E}$. Moreover every bounded interval or singleton set belonging to the collection $\mathcal{C}$ belongs to at least one of the collections $\mathcal{C}_k$, and therefore belongs to exactly one of the collections $\mathcal{D}_k$, where $\mathcal{D}_k = \mathcal{C}_k \setminus \bigcup_{j<k} \mathcal{C}_j$. It follows that

$$\begin{aligned}
\mu_F^*\left(\bigcup_{E\in\mathcal{E}} E\right) &\leq \sum_{J\in\mathcal{C}} m_F(J) = \sum_{k\in K}\sum_{J\in\mathcal{D}_k} m_F(J) \\
&\leq \sum_{k\in K}\sum_{J\in\mathcal{C}_k} m_F(J) \leq \sum_{k\in K}\left(\mu_F^*(E_k) + \frac{\varepsilon}{2^k}\right) \\
&\leq \sum_{k\in K} \mu_F^*(E_k) + \varepsilon
\end{aligned}$$

Thus $\mu_F^*\left(\bigcup_{E\in\mathcal{E}} E\right) \leq \sum_{k\in K} \mu_F^*(E_k) + \varepsilon$, no matter how small the value of ε. It follows that $\mu_F^*\left(\bigcup_{E\in\mathcal{E}} E\right) \leq \sum_{k\in K} \mu_F^*(E_k)$, as required. ∎

Proposition 10.14

Let $F \colon \mathbb{R} \to \mathbb{R}$ be a non-decreasing function of a real-variable and, for any singleton set or bounded interval K, let $m_F(K)$ denote the Stieltjes content of K with respect to the function F. Let J be a bounded interval or singleton set in $\mathbb{R}$. Then

$$\mu_F^*(A) = \mu_F^*(A \cap J) + \mu_F^*(A \setminus J)$$

for all subsets A of $\mathbb{R}$.

Proof

First we deal with the case when $\mu_F^*(A) = +\infty$, and this case either $\mu_F^*(A \cap J) = +\infty$ or else $\mu_F^*(A \setminus J) = +\infty$ because otherwise the subadditivity of Lebesgue-Stieltjes outer measure (Proposition 10.13) would ensure that $\mu_F^*(A)$, being non-negative and less than the sum of two finite quantities, would itself be a finite quantity. The stated result is thus valid in cases where $\mu_F^*(A) = +\infty$.

Now suppose that $\mu_F^*(A) < +\infty$. Let some positive real number ε be given. It then follows from the definition of Lebesgue-Stieltjes outer measure that there exists a collection $(C_i : i \in I)$ of sets, made up of bounded intervals and singleton sets, which is indexed by a countable set I, and for which

$$\sum_{i \in I} m_F(C_i) < \mu_F^*(A) + \varepsilon.$$

Then, for each $i \in I$, Proposition 7.4 guarantees the existence of a finite list $D_{i,1}, D_{i,2}, \ldots D_{i,q(i)}$ of sets, made up of bounded intervals and singleton sets, satisfying the following conditions:

- the sets $D_{i,1}, D_{i,2}, \ldots D_{i,q(i)}$ are pairwise disjoint;
- C_i is the union of all the sets $D_{i,k}$ for which $1 \leq k \leq q(i)$;
- $C_i \cap J$ is the union of those sets $D_{i,k}$ with $1 \leq k \leq q(i)$ for which $D_{i,k} \subset C_i \cap J$.

For each $i \in I$, let $L(i)$ denote the set of integers between 1 and $q(i)$ for which $D_{i,k} \not\subset C_i \cap J$. and let I_0 denote the subset of I consisting of those $i \in I$ for which $L(i)$ is non-empty. Then

$$C_i \setminus J \subset \bigcup_{k \in L(i)} D_{i,k}$$

for all $i \in I_0$, and

$$A \setminus J \subset \bigcup_{i \in I_0} (C_i \setminus J),$$

and therefore

$$A \setminus J \subset \bigcup_{i \in I_0} \bigcup_{k \in L(i)} D_{i,k}$$

It then follows from the definition of Lebesgue-Stieltjes outer measure that

$$\mu_F^*(A \setminus J) \leq \sum_{i \in I_0} \sum_{k \in L(i)} m_F(D_{i,k}),$$

where $m_F(D_{i,k})$ denotes the content of the set $D_{i,k}$ for all $i \in I$ and for all integers k in the range $1 \leq k \leq q(i)$.

But, for each $i \in I_0$, the content $m_F(C_i)$ of the set C_i is equal to the sum of the contents $m_F(D_{i,k})$ of the sets $D_{i,k}$ for all integer values of k satisfying $1 \leq k \leq q(i)$ (see Proposition 7.3), whilst the content $m_F(C_i \cap J)$ of the set $C_i \cap J$ is equal to the sum of the contents $m_F(D_{i,k})$ of those sets $D_{i,k}$ with $1 \leq k \leq q(i)$ for which $D_{i,k} \subset C_i \cap J$. It follows that, for all $i \in I_0$,

$$\sum_{k \in L(i)} m_F(D_{i,k}) = m_F(C_i) - m_F(C_i \cap J).$$

Also $m_F(C_i) = m_F(C_i \cap J)$ for all $i \in I \setminus I_0$. It follows that

$$\begin{aligned}
\mu_F^*(A \setminus J) &\leq \sum_{i \in I_0} \sum_{k \in L(i)} m_F(D_{i,k}) \\
&= \sum_{i \in I_0} (m_F(C_i) - m_F(C_i \cap J)) \\
&= \sum_{i \in I} (m_F(C_i) - m_F(C_i \cap J)).
\end{aligned}$$

The definition of definition of Lebesgue-Stieltjes outer measure also ensures that

$$\mu_F^*(A \cap J) \leq \sum_{i \in I} m_F(C_i \cap J).$$

Adding these two inequalities, we find that

$$\mu_F^*(A \cap J) + \mu_F^*(A \setminus J) \leq \sum_{i \in I} \mu(C_i) < \mu_F^*(A) + \varepsilon.$$

10. Stieltjes Measure (continued)

We have now shown that

$$\mu_F^*(A \cap J) + \mu_F^*(A \setminus J) < \mu_F^*(A) + \varepsilon$$

for all strictly positive numbers ε. It follows that

$$\mu_F^*(A \cap J) + \mu_F^*(A \setminus J) \leq \mu_F^*(A).$$

The reverse inequality

$$\mu_F^*(A) \leq \mu_F^*(A \cap J) + \mu_F^*(A \setminus J),$$

is a consequence of Proposition 10.13. It follows that

$$\mu_F^*(A) = \mu_F^*(A \cap J) + \mu_F^*(A \setminus J),$$

as required. $\blacksquare$