MAU22200—Advanced Analyis
School of Mathematics, Trinity College
Hilary Term 2020
Section 5: Some Properties of Infinite Sequences and Series

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5. Some Properties of Infinite Sequences and Series

5.1. Least Upper Bounds and Greatest Lower Bounds

Definition

Let S be a set of real numbers which is bounded above. The *least upper bound*, or *supremum*, of the set S is the smallest real number that is greater than or equal to elements of the set S, and is denoted by $\sup S$.

Thus if S is a set of real numbers that is bounded above, then the least upper bound $\sup S$ of the set S is characterized by the following two properties:

- for all $x \in S$, $x \le \sup S$;
- if u is a real number, and if, for all $x \in S$, $x \le u$ then $\sup S \le u$.

The *Least Upper Bound Property* of the real number system guarantees that, given any non-empty set S of real numbers that is bounded above, there exists a least upper bound $\sup S$ for the set S.

Definition

Let S be a set of real numbers which is bounded below. The *greatest lower bound*, or *infimum*, of the set S is the largest real number that is less than or equal to elements of the set S, and is denoted by $\inf S$.

Thus if S is a set of real numbers that is bounded below, then the greatest lower bound $\inf S$ of the set S is characterized by the following two properties:

- for all $x \in S$, $x \geq \inf S$;
- if l is a real number, and if, for all $x \in S$, $x \geq l$ then $\inf S \geq l$.

Given any non-empty set S of real numbers that is bounded below, there exists a greatest lower bound $\inf S$ for the set S.

5.2. Monotonic Sequences

An infinite sequence $x_1, x_2, x_3, \ldots$ of real numbers is said to be *strictly increasing* if $x_{j+1} > x_j$ for all positive integers j, *strictly decreasing* if $x_{j+1} < x_j$ for all positive integers j, *non-decreasing* if $x_{j+1} \geq x_j$ for all positive integers j, *non-increasing* if $x_{j+1} \leq x_j$ for all positive integers j. A sequence satisfying any one of these conditions is said to be *monotonic*; thus a monotonic sequence is either non-decreasing or non-increasing.

Theorem 5.1

Any non-decreasing sequence of real numbers that is bounded above is convergent. Similarly any non-increasing sequence of real numbers that is bounded below is convergent.

Proof

Let $x_1, x_2, x_3, \ldots$ be a non-decreasing sequence of real numbers that is bounded above. It follows from the Least Upper Bound Axiom that there exists a least upper bound p for the set $\{x_j : j \in \mathbb{N}\}$. We claim that the sequence converges to p.

Let some strictly positive real number ε be given. We must show that there exists some positive integer N such that $|x_j - p| < \varepsilon$ whenever $j \geq N$. Now $p - \varepsilon$ is not an upper bound for the set $\{x_j : j \in \mathbb{N}\}$ (since p is the least upper bound), and therefore there must exist some positive integer N such that $x_N > p - \varepsilon$. But then $p - \varepsilon < x_j \leq p$ whenever $j \geq N$, since the sequence is non-decreasing and bounded above by p. Thus $|x_j - p| < \varepsilon$ whenever $j \geq N$. Therefore $x_j \to p$ as $j \to +\infty$, as required.

If the sequence $x_1, x_2, x_3, \ldots$ is non-increasing and bounded below then the sequence $-x_1, -x_2, -x_3, \ldots$ is non-decreasing and bounded above, and is therefore convergent. It follows that the sequence $x_1, x_2, x_3, \ldots$ is also convergent. $\blacksquare$

5.3. Upper and Lower Limits

Let $a_1, a_2, a_3, \ldots$ be a bounded infinite sequence of real numbers, and, for each positive integer j, let

$$S_j = \{a_j, a_{j+1}, a_{j+2}, \ldots\} = \{a_k : k \geq j\}.$$

The sets $S_1, S_2, S_3, \ldots$ are all bounded. It follows that there exist well-defined infinite sequences $u_1, u_2, u_3, \ldots$ and $l_1, l_2, l_3, \ldots$ of real numbers, where $u_j = \sup S_j$ and $l_j = \inf S_j$ for all positive integers j. Now S_{j+1} is a subset of S_j for each positive integer j, and therefore $u_{j+1} \leq u_j$ and $l_{j+1} \geq l_j$ for each positive integer j. It follows that the bounded infinite sequence $(u_j : j \in \mathbb{N})$ is a non-increasing sequence, and is therefore convergent (Theorem 5.1). Similarly the bounded infinite sequence $(l_j : j \in \mathbb{N})$ is a non-decreasing sequence, and is therefore convergent.

We define

$$\limsup_{j \to +\infty} a_j = \lim_{j \to +\infty} u_j = \lim_{j \to +\infty} \sup\{a_j, a_{j+1}, a_{j+2}, \ldots\}$$

and

$$\liminf_{j \to +\infty} a_j = \lim_{j \to +\infty} l_j = \lim_{j \to +\infty} \inf\{a_j, a_{j+1}, a_{j+2}, \ldots\}.$$

The quantity $\limsup_{j \to +\infty} a_j$ is referred to as the *upper limit* of the sequence $a_1, a_2, a_3, \ldots$. The quantity $\liminf_{j \to +\infty} a_j$ is referred to as the *lower limit* of the sequence $a_1, a_2, a_3, \ldots$.

Note that every bounded infinite sequence $a_1, a_2, a_3, \ldots$ of real numbers has a well-defined upper limit $\limsup_{j \to +\infty} a_j$ and a well-defined lower limit $\liminf_{j \to +\infty} a_j$.

Proposition 5.2

A bounded infinite sequence $a_1, a_2, a_3, \ldots$. of real numbers is convergent if and only if $\liminf_{j \to +\infty} a_j = \limsup_{j \to +\infty} a_j$, in which case the limit of the sequence is equal to the common value of its upper and lower limits.

Proof

For each positive integer j, let $u_j = \sup S_j$ and $l_j = \inf S_j$, where

$$S_j = \{a_j, a_{j+1}, a_{j+2}, \ldots\} = \{a_k : k \geq j\}.$$

Then $\liminf_{j \to +\infty} a_j = \lim_{j \to +\infty} l_j$ and $\limsup_{j \to +\infty} a_j = \lim_{j \to +\infty} u_j$.

Suppose that $\liminf_{j \to +\infty} a_j = \limsup_{j \to +\infty} a_j = c$ for some real number c. Then, given any positive real number ε, there exist positive integers N_1 and N_2 such that $c - \varepsilon < l_j \leq c$ whenever $j \geq N_1$, and $c \leq u_j < c + \varepsilon$ whenever $j \geq N_2$. Let N be the maximum of N_1 and N_2. If $j \geq N$ then $a_j \in S_N$, and therefore

$$c - \varepsilon < l_N \leq a_j \leq u_N < c + \varepsilon.$$

Thus $|a_j - c| < \varepsilon$ whenever $j \geq N$. This proves that the infinite sequence $a_1, a_2, a_3, \ldots$ converges to the limit c.

Conversely let $a_1, a_2, a_3, \ldots$ be a bounded sequence of real numbers that converges to some value c. Let $\varepsilon > 0$ be given. Then there exists some positive integer N such that $c - \frac{1}{2}\varepsilon < a_j < c + \frac{1}{2}\varepsilon$ whenever $j \geq N$. It follows that $S_j \subset (c - \frac{1}{2}\varepsilon, c + \frac{1}{2}\varepsilon)$ whenever $j \geq N$. But then

$$c - \tfrac{1}{2}\varepsilon \leq l_j \leq u_j \leq c + \tfrac{1}{2}\varepsilon$$

whenever $j \geq N$, where $u_j = \sup S_j$ and $l_j = \inf S_j$. We see from this that, given any positive real number ε, there exists some positive integer N such that $|l_j - c| < \varepsilon$ and $|u_j - c| < \varepsilon$ whenever $j \geq N$. It follows from this that

$$\limsup_{j \to +\infty} a_j = \lim_{j \to +\infty} u_j = c \text{ and } \liminf_{j \to +\infty} a_j = \lim_{j \to +\infty} l_j = c,$$

as required. ∎

5.4. Rearrangement of Infinite Series

Example

Consider the infinite series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots .$$

For each positive integer k, let p_k denote the kth partial sum of this infinite series, defined so that

$$p_k = \sum_{j=1}^{k} (-1)^{j-1} \frac{1}{j}.$$

The absolute values of the summands constitute a decreasing sequence, and accordingly examination of the form of the infinite series establishes that

$$p_1 > p_3 > p_5 > p_7 > \cdots$$

$$p_2 < p_4 < p_6 < p_8 < \cdots$$

Moreover $p_{2m} \leq p_{2m+1} \leq p_1$ and $p_{2m+1} \geq p_{2m} \geq p_2$ for all positive integers m. Thus p_1, p_3, p_5, p_7 is a decreasing sequence that is bounded below, and p_2, p_4, p_6, p_8 is an increasing sequence that is bounded above. A standard result of real analysis ensures that these bounded monotonic sequences are convergent. Moreover

$$\begin{aligned}
\lim_{m\to+\infty} p_{2m+1} &= \lim_{m\to\infty}\left(p_{2m} + \frac{1}{2m+1}\right) \\
&= \lim_{m\to\infty} p_{2m} + \lim_{m\to+\infty}\frac{1}{2m+1} \\
&= \lim_{m\to\infty} p_{2m}.
\end{aligned}$$

It then follows easily from examination of the definition of convergence that the infinite sequence $p_1, p_2, p_3, \ldots$ converges, and

$$\lim_{j \to +\infty} p_j = \lim_{m \to +\infty} p_{2m} = \lim_{m \to +\infty} p_{2m+1}.$$

Let $\alpha = \lim_{j \to +\infty} p_j$. Then $p_2 < \alpha < p_1$, and thus $\frac{1}{2} < \alpha < 1$.

Now consider the infinite series

$$1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \frac{1}{12} + \cdots .$$

The individual summands are those of the infinite series previously considered, but they occur in a different order. This new infinite series is thus a *rearrangement* of the infinite series previously considered. Nevertheless the sum of this new infinite series may be represented as

$$\left(1 - \frac{1}{2}\right) - \frac{1}{4} + \left(\frac{1}{3} - \frac{1}{6}\right) - \frac{1}{8} + \left(\frac{1}{5} - \frac{1}{10}\right) - \frac{1}{12} + \cdots$$

and therefore the sum of the new infinite series is equal to that of the infinite series

$$\frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \frac{1}{12} + \cdots ,$$

and is therefore equal to $\frac{1}{2}\alpha$. This example demonstrates that when the terms of an infinite series are rearranged, so that they are summed together in a different order, the sum of the rearranged series is not necessarily equal to that of the original series.

The example just discussed considers the behaviour of a particular infinite series that is convergent but not *absolutely convergent*. An infinite series $\sum_{j=1}^{+\infty} a_j$ is said to be *absolutely convergent* if $\sum_{j=1}^{+\infty} |a_j|$ is convergent. The following proposition and its corollaries ensure that any absolutely convergent infinite series may be rearranged at will without affecting convergence, and without changing the value of the sum of the series. In particular an infinite series whose summands are non-negative may be rearranged without affecting the value of the sum of that infinite series.

Proposition 5.3

Let $\sum_{j=1}^{+\infty} a_j$ *be a convergent infinite series, where* a_j *is real and* $a_j \geq 0$ *for all positive integers* j. *Let* Q *be the subset of the real numbers consisting of the values of all sums of the form* $\sum_{j \in F} a_j$ *obtained as* F *ranges over all the non-empty finite subsets of* $\mathbb{N}$. *Then*

$$\sum_{j=1}^{+\infty} a_j = \sup Q.$$

Proof

For each positive integer k, let

$$p_k = \sum_{j=1}^{k} a_j.$$

This number p_k is referred to as the kth *partial sum* of the infinite series $a_1 + a_2 + a_3 + \cdots$. The definition of the sum of this infinite series then ensures that

$$\sum_{j=1}^{+\infty} a_j = \lim_{k \to +\infty} p_k.$$

Moreover $p_1 \leq p_2 \leq p_3 \leq \cdots$, because $a_j \geq 0$ for all positive integers j, and therefore

$$\sum_{j=1}^{+\infty} a_j = \sup\{p_k : k \in \mathbb{N}\}.$$

For each non-empty finite subset F of the set $\mathbb{N}$ of positive integers, let

$$q_F = \sum_{j \in F} a_j.$$

If F and H are finite subsets of $\mathbb{N}$, and if $F \subset H$ then $q_F \leq q_H$, because the summand a_j is non-negative for all positive integers j.

Now, given any non-empty finite subset F of $\mathbb{N}$ there exists some positive integer k such that $F \subset J_k$, where $J_k = \{1, 2, \ldots, k\}$. But then

$$q_F \leq q_{J_k} = p_k \leq \sum_{j=1}^{+\infty} a_j.$$

Therefore the set Q consisting of the values of the sums q_F as F ranges over all the non-empty finite subsets F of $\mathbb{N}$ is bounded above. Moreover it is non-empty. The Least Upper Bound Principle then ensures that the set Q has a well-defined least upper bound $\sup Q$.

Let $s = \sup Q$. We have shown that $q_F \leq \sum_{j=1}^{+\infty} a_j$ for each non-empty finite subset F of $\mathbb{N}$. It follows that $s \leq \sum_{j=1}^{+\infty} a_j$. But $p_k \in Q$ for all positive integers k, because $p_k = q_{J_k}$, and therefore $p_k \leq s$. Taking limits as $k \to +\infty$, we find that

$$\sum_{j=1}^{+\infty} a_j = \lim_{k \to +\infty} p_k \leq s.$$

The inequalities just obtained together ensure that

$$\sum_{j=1}^{+\infty} a_j = s = \sup Q,$$

as required. ∎

A *permutation* of the set $\mathbb{N}$ of positive integers is a function $\sigma\colon \mathbb{N} \to \mathbb{N}$ from the set $\mathbb{N}$ to itself that is bijective. A function $\sigma\colon \mathbb{N} \to \mathbb{N}$ is thus a permutation if and only if it has a well-defined inverse $\sigma^{-1}\colon \mathbb{N} \to \mathbb{N}$. This is the case if and only if, given any positive integer k, there exists a unique positive integer j for which $k = \sigma(j)$.

Definition

An infinite sequence $b_1, b_2, b_3, \ldots$ of real numbers is said to be a *rearrangement* of an infinite sequence $a_1, a_2, a_3, \ldots$ if there exists a permutation σ of the set $\mathbb{N}$ of positive integers such that $b_k = a_{\sigma(k)}$ for all positive integers k. In this situation we also say that the infinite series $\sum\limits_{k=1}^{+\infty} b_k$ is a rearrangement of the infinite series $\sum\limits_{j=1}^{+\infty} a_j$.

Corollary 5.4

Let $\sum_{j=1}^{+\infty} a_j$ *be a convergent infinite series, and let* $\sum_{k=1}^{+\infty} b_k$ *be a rearrangement of infinite series* $\sum_{j=1}^{+\infty} a_j$. *Suppose that* $a_j \geq 0$ *for all positive integers* j. *Then the infinite series* $\sum_{k=1}^{+\infty} b_k$ *is convergent, and* $\sum_{k=1}^{+\infty} b_k = \sum_{j=1}^{+\infty} a_j$.

Proof

There exists a permutation $\sigma\colon \mathbb{N} \to \mathbb{N}$ of the set $\mathbb{N}$ of positive integers such that $b_k = a_{\sigma(k)}$ for all positive integers k. Let $q_F = \sum_{j \in F} a_j$ for all non-empty finite subsets F of $\mathbb{N}$, and let $r_G = \sum_{k \in G} b_k$ for all non-empty finite subsets G of $\mathbb{N}$. Then

$$q_{\sigma(G)} = \sum_{j \in \sigma(G)} a_j = \sum_{k \in G} a_{\sigma(k)} = \sum_{k \in G} b_k = r_G$$

for all non-empty finite subsets G of $\mathbb{N}$, and accordingly $q_F = r_{\sigma^{-1}(F)}$ for all non-empty finite subsets F of $\mathbb{N}$. Moroever G is a non-empty finite subset of $\mathbb{N}$ if and only if $\sigma(G)$ is a non-empty finite subset of $\mathbb{N}$.

It follows that the set Q consisting of all sums of the form q_F as F ranges over the non-empty finite subsets of $\mathbb{N}$ is also the set consisting of all sums of the form r_G as G ranges over the non-empty finite subsets of $\mathbb{N}$. It follows from Proposition 5.3 that

$$\sum_{j=1}^{+\infty} a_j = \sup Q = \sum_{k=1}^{+\infty} b_k,$$

as required. ∎

It follows from Corollary 5.4 that, given any collection $(c_\alpha : \alpha \in A)$ of *non-negative* real numbers c_α indexed by the members of a countable set A, we can form the sum $\sum_{\alpha \in A} c_\alpha$. If the countable indexing set A is infinite then there exists an infinite sequence $\alpha_1, \alpha_2, \alpha_3, \ldots$ in which each element of the set A occurs exactly once. Then

$$\sum_{\alpha \in A} c_\alpha = \sum_{j=1}^{+\infty} c_{\alpha_j}.$$

The requirement that $c_\alpha \geq 0$ for all $\alpha \in A$ ensures that the value of $\sum_{j=1}^{+\infty} c_{\alpha_j}$ does not depend on the choice of infinite sequence $\alpha_1, \alpha_2, \alpha_3, \ldots$ enumerating the elements of the indexing set A.

Let $c_1, c_2, c_3, \ldots$ be an infinite sequence of real numbers that are not necessarily all non-negative or all non-positive, and let $c_j^+ = \max(c_j, 0)$ and $c_j^- = \min(0, c_j)$ for all positive integers j. Then $c_j^+ \geq 0$, $c_j^- \leq 0$, $c_j = c_j^+ + c_j^-$ and $|c_j| = c_j^+ - c_j^- = c_j^+ + |c_j^-|$ for all positive integers j. Moreover, for each positive integer j, at most one of the numbers c_j^+ asnd c_j^- is non-zero. Now $0 \leq c_j^+ \leq |c_j|$ and $0 \leq |c_j^-| \leq |c_j|$ for all positive integers j. It follows from this that $\sum_{j=1}^{+\infty} |c_j|$ is convergent if and only if both $\sum_{j=1}^{+\infty} c_j^+$ and $\sum_{j=1}^{+\infty} c_j^-$ convergent. In this case we say that the infinite series $\sum_{j=1}^{+\infty} c_j$ is *absolutely convergent*.

Corollary 5.5

Let $\sum_{j=1}^{+\infty} a_j$ *be an absolutely convergent infinite series, and let* $\sum_{k=1}^{+\infty} b_k$ *be a rearrangement of infinite series* $\sum_{j=1}^{+\infty} a_j$*. Then the infinite series* $\sum_{k=1}^{+\infty} b_k$ *is absolutely convergent, and* $\sum_{k=1}^{+\infty} b_k = \sum_{j=1}^{+\infty} a_j$*.*

Proof

There exists a permutation $\sigma\colon \mathbb{N} \to \mathbb{N}$ of the set $\mathbb{N}$ of positive integers with the property that $b_k = a_{\sigma(k)}$ for all positive integers k. Let $a_j^+ = \max(a_j, 0)$ and $a_j^- = \min(0, a_j)$ for all positive integers j and $b_k^+ = \max(b_k, 0)$ and $b_k^- = \min(0, b_k)$ for all positive integers k. The absolute convergence of $\sum_{j=1}^{\infty} a_j$ then ensures that the infinite series $\sum_{j=1}^{\infty} a_j^+$ and $\sum_{j=1}^{\infty} a_j^-$ both converge. It then follows from Corollary 5.4 that

$$\sum_{j=1}^{+\infty} |a_j| = \sum_{j=1}^{+\infty} a_j^+ - \sum_{j=1}^{+\infty} a_j^- = \sum_{k=1}^{+\infty} b_k^+ - \sum_{k=1}^{+\infty} b_k^- = \sum_{k=1}^{+\infty} |b_k|$$

and

$$\sum_{j=1}^{+\infty} a_j = \sum_{j=1}^{+\infty} a_j^+ + \sum_{j=1}^{+\infty} a_j^- = \sum_{k=1}^{+\infty} b_k^+ + \sum_{k=1}^{+\infty} b_k^- = \sum_{k=1}^{+\infty} b_k.$$

The result follows. ∎

5.5. The Extended Real Number System

It is sometimes convenient to make use of the *extended real line* $[-\infty, +\infty]$. This is the set $\mathbb{R} \cup \{-\infty, +\infty\}$ obtained on adjoining to the real line $\mathbb{R}$ two extra elements $+\infty$ and $-\infty$ that represent points at 'positive infinity' and 'negative infinity' respectively. We define

$$c + (+\infty) = (+\infty) + c = +\infty$$

and

$$c + (-\infty) = (-\infty) + c = -\infty$$

for all real numbers c.

We also define products of non-zero real numbers with these extra elements $\pm\infty$ so that

$$\begin{aligned} c \times (+\infty) &= (+\infty) \times c = +\infty \quad \text{when } c > 0, \\ c \times (-\infty) &= (-\infty) \times c = -\infty \quad \text{when } c > 0, \\ c \times (+\infty) &= (+\infty) \times c = -\infty \quad \text{when } c < 0, \\ c \times (-\infty) &= (-\infty) \times c = +\infty \quad \text{when } c < 0, \end{aligned}$$

We also define

$$0 \times (+\infty) = (+\infty) \times 0 = 0 \times (-\infty) = (-\infty) \times 0 = 0,$$

and

$$(+\infty) \times (+\infty) = (-\infty) \times (-\infty) = +\infty,$$

$$(+\infty) \times (-\infty) = (-\infty) \times (+\infty) = -\infty.$$

The sum of $+\infty$ and $-\infty$ is not defined.

We define $-(+\infty) = -\infty$ and $-(-\infty) = +\infty$. The difference $p - q$ of two extended real numbers is then defined by the formula $p - q = p + (-q)$, unless $p = q = +\infty$ or $p = q = -\infty$, in which cases the difference of the extended real numbers p and q is not defined.

We extend the definition of inequalities to the extended real line in the obvious fashion, so that $c < +\infty$ and $c > -\infty$ for all real numbers c, and $-\infty < +\infty$.

Given any real number c, we define

$$\begin{aligned} [c, +\infty] &= [c, +\infty) \cup \{+\infty\} = \{p \in [-\infty, \infty] : p \geq c\}, \\ (c, +\infty] &= (c, +\infty) \cup \{+\infty\} = \{p \in [-\infty, \infty] : p > c\}, \\ [-\infty, c] &= (-\infty, c] \cup \{-\infty\} = \{p \in [-\infty, \infty] : p \leq c\}, \\ [-\infty, c) &= (-\infty, c) \cup \{-\infty\} = \{p \in [-\infty, \infty] : p < c\}. \end{aligned}$$

There is an order-preserving bijective function $\varphi\colon [-\infty, +\infty] \to [-1, 1]$ from the extended real line $[-\infty, +\infty]$ to the closed interval $[-1, 1]$ which is defined such that $\varphi(+\infty) = 1$, $\varphi(-\infty) = -1$, and $\varphi(c) = \dfrac{c}{1 + |c|}$ for all real numbers c. Let us define $\rho(p, q) = |\varphi(q) - \varphi(p)|$ for all extended real numbers p and q. Then the set $[-\infty, +\infty]$ becomes a metric space with distance function ρ. Moreover the function $\varphi\colon [-\infty, +\infty] \to [-1, 1]$ is a homeomorphism from this metric space to the closed interval $[-1, 1]$. It follows directly from this that $[-\infty, +\infty]$ is a compact metric space. Moreover an infinite sequence $(p_j : j \in \mathbb{N})$ of extended real numbers is convergent if and only if the corresponding sequence $(\varphi(p_j) : j \in \mathbb{N})$ of real numbers is convergent.

5. Some Properties of Infinite Sequences and Series (continued)

Given any non-empty set S of extended real numbers, we can define $\sup S$ to be the least extended real number p with the property that $s \leq p$ for all $s \in S$. If the set S does not contain the extended real number $+\infty$, and if there exists some real number B such that $s \leq B$ for all $s \in S$, then $\sup S < +\infty$; otherwise $\sup S = +\infty$. Similarly we define $\inf S$ to be the greatest extended real number p with the property that $s \geq p$ for all $s \in S$. If the set S does not contain the extended real number $-\infty$, and if there exists some real number A such that $s \geq A$ for all $s \in S$, then $\inf S > +\infty$; otherwise $\inf S = -\infty$. Moreover

$$\varphi(\sup S) = \sup \varphi(S) \text{ and } \varphi(\inf S) = \inf \varphi(S),$$

where $\varphi\colon [-\infty, +\infty] \to [-1, 1]$ is the homeomorphism defined such that $\varphi(+\infty) = 1$, $\varphi(-\infty) = -1$ and $\varphi(c) = c(1 + |c|)^{-1}$ for all real numbers c.

Given any sequence $(p_j : j \in \mathbb{N})$ of extended real numbers, we define the *upper limit* $\limsup_{j\to+\infty} p_j$ and the *lower limit* $\liminf_{j\to+\infty} p_j$ of the sequence so that

$$\limsup_{j\to+\infty} p_j = \lim_{j\to+\infty} \sup\{p_k : k \geq j\}$$

and

$$\liminf_{j\to+\infty} p_j = \lim_{j\to+\infty} \inf\{p_k : k \geq j\}.$$

Every sequence of extended real numbers has both an upper limit and a lower limit. Moreover an infinite sequence of extended real numbers converges to some extended real number if and only if the upper and lower limits of the sequence are equal. (These results follow easily from the corresponding results for bounded sequences of real numbers, on using the identities

$$\varphi(\limsup_{j \to +\infty} p_j) = \limsup_{j \to +\infty} \varphi(p_j), \quad \varphi(\liminf_{j \to +\infty} p_j) = \liminf_{j \to +\infty} \varphi(p_j),$$

where $\varphi\colon [-\infty, +\infty] \to [-1, 1]$ is the homeomorphism defined above.)

The function that sends a pair (p, q) of extended real numbers to the extended real number $p + q$ is not defined when $p = +\infty$ and $q = -\infty$, or when $p = -\infty$ and $q = +\infty$ but is continuous elsewhere. The function that sends a pair (p, q) of extended real numbers to the extended real number pq is defined everywhere. This function is discontinuous when $p = \pm\infty$ and $q = 0$, and when $p = 0$ and $q = \pm\infty$. It is continuous for all other values of the extended real numbers p and q.

Let $a_1, a_2, a_3, \ldots$ be an infinite sequence of extended real numbers which does not include both the values $+\infty$ and $-\infty$, and let $p_k = \sum_{j=0}^{k} a_j$ for all natural numbers k. If the infinite sequence $p_1, p_2, p_3, \ldots$ of extended real numbers converges in the extended real line $[-\infty, +\infty]$ to some extended real number p, then this value p is said to be the *sum* of the infinite series $\sum\limits_{j=1}^{+\infty} a_j$, and we write $\sum\limits_{j=1}^{+\infty} a_j = p$.

It follows easily from this definition that if $+\infty$ is one of the values of the infinite series $a_1, a_2, a_3, \ldots$, then $\sum_{j=1}^{+\infty} a_j = +\infty$. Similarly if $-\infty$ is one of the values of this infinite series then then $\sum_{j=1}^{+\infty} a_j = -\infty$. Suppose that the members of the sequence $a_1, a_2, a_3, \ldots$ are all real numbers. Then $\sum_{j=1}^{+\infty} a_n = +\infty$ if and only if, given any real number B, there exists some real number N such that $\sum_{j=1}^{k} a_n > B$ whenever $k \geq N$. Similarly $\sum_{j=1}^{+\infty} a_j = -\infty$ if and only if, given any real number A, there exists some real number N such that $\sum_{j=1}^{k} a_j < A$ whenever $k \geq N$.