MAU22200—Advanced Analyis
School of Mathematics, Trinity College
Hilary Term 2020
Section 9: Modes of Convergence on Measure Spaces

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9. Modes of Convergence on Measure Spaces

9.1. Egorov's Theorem

Proposition 9.1

Let $(X, \mathcal{A}, \mu)$ be a measure space, and let E be a measurable subset of X, where $\mu(E) < +\infty$. Let $f_1, f_2, f_3, \ldots$ be an infinite sequence of measurable real-valued functions that converges on E to a measurable function f. Then, given any strictly positive real numbers ε and δ, there exists a measurable subset F of E and a positive integer N such that $\mu(F) < \delta$ and $|f_j(x) - f(x)| < \varepsilon$ whenever $x \in E \setminus F$ and $j \geq N$.

Proof

Let strictly positive real numbers ε and δ be given and, for each positive integer k, let

$$E_k = \bigcap_{j=k}^{+\infty} \{x \in X : |f_j(x) - f(x)| < \varepsilon\}.$$

Let $x \in E$. Then $\lim_{j \to +\infty} f_j(x) = f(x)$, and therefore there exists some positive integer k such that $|f_j(x) - f(x)| < \varepsilon$ whenever $j \geq k$. But then $x \in E_k$. It follows from this that $E = \bigcup_{k=1}^{+\infty} E_k$.

Now $E_j \subset E_{j+1}$ for all positive integers j. It follows from the countable additivity of the measure μ that $\mu(E) = \lim_{k \to +\infty} \mu(E_k)$ (see Lemma 7.25). Now $\mu(E) < +\infty$. It follows that there exists some positive integer N that is large enough to ensure that $\mu(E_N) > \mu(E) - \delta$. Let $F = E \setminus E_N$. Then $\mu(F) < \delta$ and $|f(x) - f_j(x)| < \varepsilon$ whenever $x \in E \setminus F$ and $j \geq N$, as required. ∎

Let X be a set, and let E be a subset of X. Let $f_1, f_2, f_3, \ldots$ be an infinite sequence of real-valued functions on X, and let f be a real-valued function on X. The sequence $f_1, f_2, f_3, \ldots$ is said to converge *uniformly* on E to the limit function f if, given any strictly positive real number ε, there exists some positive integer N (independent of the choice of x in the set E) such that $|f_j(x) - f(x)| < \varepsilon$ whenever $x \in E$ and $j \geq N$.

Sequences of functions that converge pointwise do not necessarily converge uniformly. An infinite sequence $f_1, f_2, f_3, \ldots$ of real-valued functions on the set X is said to converge *pointwise* to the real-valued function f on a subset E of X if, given any strictly positive real number ε, and given any point x of E, there exists some positive integer $N(x)$ (in general dependent on the choice of both ε and x, such set E) such that $|f_j(x) - f(x)| < \varepsilon$ whenever $j \geq N(x)$.

Now let $(X, \mathcal{A}, \mu)$ be a measure space, let $f_1, f_2, f_3, \ldots$ be an infinite sequence of measurable real-valued functions on X, and let f be a measurable real-valued function on f. If there exists a subset F of X satisfying $\mu(F) = 0$, and if the infinite sequence $f_1, f_2, f_3, \ldots$ converges pointwise to f on the complement $X \setminus F$ of F in X, then the sequence $f_1, f_2, f_3, \ldots$ is said to converge *pointwise almost everywhere* on X.

Theorem 9.2 (Egorov's Theorem)

Let $(X, \mathcal{A}, \mu)$ be a measure space, and let E be a measurable subset of X, where $\mu(E) < +\infty$. Let $f_1, f_2, f_3, \ldots$ be an infinite sequence of measurable real-valued functions that converges pointwise almost everywhere on E to a measurable function f. Then, given any strictly positive real number δ, there exists a subset F of E such that $\mu(F) < \delta$ and the sequence $f_1, f_2, f_3, \ldots$ of real-valued functions converges uniformly to the limit function f on $E \setminus F$.

Proof

If the infinite sequence $f_1, f_2, f_3, \ldots$ does not converge to f throughout the set E, then there exists some subset E_0 of E such that $\mu(E \setminus E_0) = 0$ and the sequence $f_1, f_2, f_3, \ldots$ converges to the limit function f at all points of E_0, because the sequence $f_1, f_2, f_3, \ldots$ does at least converge to f almost everywhere on E. If then, for a given strictly positive real number δ, we show the existence of a subset F_0 of E_0 such that $\mu(F_0) < \delta$ and the sequence $f_1, f_2, f_3, \ldots$ converges uniformly to f on $E_0 \setminus F_0$, and if we take $F = F_0 \cup (E \setminus E_0)$, then $\mu(F) < \delta$ and the sequence $f_1, f_2, f_3, \ldots$ of functions converges uniformly to the limit function f on $E \setminus F$. Thus we may assume, without loss of generality, that the sequence of functions $f_1, f_2, f_3, \ldots$ converges to the limit function throughout the set E.

Thus suppose that the sequence $f_1, f_2, f_3, \ldots$ converges pointwise to the limit function f throughout E, and let some strictly positive real number δ be given. It follows on applying Proposition 9.1, that there exist positive integers $N_1, N_2, N_3, \ldots$ and measurable subsets $F_1, F_2, F_3 \ldots.$ of E such that $\mu(F_k) < 2^{-k}\delta$ and $|f_j(x) - f(x)| < 1/k$ whenever $x \in E \setminus F_k$ and $j \geq N_k$. Let $F = \bigcup_{k=1}^{+\infty} F_k$. Then

$$\mu(F) \leq \sum_{k=1}^{+\infty} \mu(F_k) < \delta.$$

The required result is thus established once we prove that the infinite sequence $f_1, f_2, f_3, \ldots$ of measurable functions converges uniformly to the limit function f on the set $E \setminus F$.

Let some strictly positive real number ε be given. Then some positive integer k can be chosen large enough to ensure that $1/k < \varepsilon$. Let $N = N_k$. Now $E \setminus F \subset E \setminus F_k$. It follows that $|f_j(x) - f(x)| < \varepsilon$ whenever $x \in E \setminus F$ and $j \geq N$. Thus the functions $f_1, f_2, f_3, \ldots$ do indeed converge uniformly to the limit function f on $E \setminus F$. The result follows. ∎

9.2. Almost Uniform Convergence

Definition

Let $(X, \mathcal{A}, \mu)$ be a measure space, let E be a measurable subset of E, let $f_1, f_2, f_3, \ldots$ be an infinite sequence of measurable real-valued functions on X, and let f be a measurable real-valued function on X. The infinite sequence $f_1, f_2, f_3, \ldots$ is said to converge *almost uniformly* on E if, given any positive real number δ, there exists a measurable subset F of E such that $\mu(F) < \delta$ and the infinite sequence $f_1, f_2, f_3, \ldots$ converges uniformly to the limit function f on $E \setminus F$.

Egorov's Theorem (Theorem 9.2) ensures that if $(X, \mathcal{A}, \mu)$ is a measure space, and if E is a measurable subset of X whose measure $\mu(E)$ is finite, then any infinite sequence of measurable real-valued functions that converges pointwise almost everywhere on E also converges almost uniformly on E.

Example

For each positive integer j, let $f_j\colon \mathbb{R} \to \mathbb{R}$ be defined so that

$$f_j(x) = \begin{cases} 1 & \text{if } j - 1 < x \leq j; \\ 0 & \text{otherwise.} \end{cases}$$

Then the infinite sequence $f_1, f_2, f_3, \ldots$ converges pointwise to the zero function throughout the real line $\mathbb{R}$ but does not converge almost uniformly with respect to Lebesgue measure on the real line.

Lemma 9.3

Let $(X, \mathcal{A}, \mu)$ be a measure space, let $f_1, f_2, f_3, \ldots$ be an infinite sequence of measurable real-valued functions on X, and let f be a measurable real-valued function on X. Suppose that the infinite sequence $f_1, f_2, f_3, \ldots$ of functions converges almost uniformly to the limit function f. Then the sequence $f_1, f_2, f_3, \ldots$ converges to f pointwise almost everywhere.

Proof

Let F be the subset of X consisting of those points x for which the infinite sequence $f_1(x), f_2(x), f_3(x), \ldots$ does not converge to $f(x)$. We must show that $\mu(F) = 0$.

Now the infinite sequence $f_1, f_2, f_3, \ldots$ converges almost uniformly on X to the limit function f, and therefore, given any strictly positive real number δ, there exists some measurable set F_δ such that $\mu(F_\delta) < \delta$ and $\lim_{j \to +\infty} f_j(x) = f(x)$ for all $x \in X \setminus F_\delta$. But then $F \subset F_\delta$, and consequently $\mu(F) \leq \mu(F_\delta) < \delta$. We conclude therefore that $\mu(F) < \delta$ for all positive real numbers δ. It follows that $\mu(F) = 0$, as required. $\blacksquare$

9.3. Convergence in Measure

Definition

Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $f_1, f_2, f_3, \ldots$ be an infinite sequence of measurable real-valued functgions on X, and let f be a measurable real-valued function on X. The sequence $f_1, f_2, f_3, \ldots$ is said to *converge in measure* to the function f if, given any strictly positive real numbers ε and δ, there exists some positive integer N such that

$$\mu\left(\{x \in X : |f_j(x) - f(x)| \geq \varepsilon\}\right) < \delta$$

whenever $j \geq N$.

The following result follows immediately from the relevant definitions.

Lemma 9.4

Let $(X, \mathcal{A}, \mu)$ be a measure space, let $f_1, f_2, f_3, \ldots$ be an infinite sequence of measurable real-valued functions on X, and let f be a measurable real-valued function on X. Then the infinite sequence $f_1, f_2, f_3, \ldots$ converges in measure to the limit function f if and only if, given any strictly positive real number ε,

$$\lim_{j \to +\infty} \mu\left(\{x \in X : |f_j(x) - f(x)| \geq \varepsilon\}\right) = 0.$$

Proposition 9.5

Let $(X, \mathcal{A}, \mu)$ be a measure space, let $f_1, f_2, f_3, \ldots$ be an infinite sequence of measurable real-valued functions on X, and let f and $\tilde{f}$ be a measurable real-valued functions on X. Suppose that the infinite sequence $f_1, f_2, f_3, \ldots$ converges in measure both to the the function fs and also to the function $\tilde{f}$. Then the functions f and $\tilde{f}$ are equal almost everywhere.

Proof

For each positive integer j and positive real number ε, let

$$\begin{aligned} E_{j,\varepsilon} &= \{x \in X : |f_j(x) - f(x)| \geq \varepsilon\}, \\ \tilde{E}_{j,\varepsilon} &= \{x \in X : |f_j(x) - \tilde{f}(x)| \geq \varepsilon\}. \end{aligned}$$

Then

$$\lim_{j \to +\infty} \mu(E_{j,\varepsilon}) \to 0 \quad \text{and} \quad \lim_{j \to +\infty} \mu(\tilde{E}_{j,\varepsilon}) \to 0.$$

Now if x is a point of X, j is a positive integer, ε is a positive real number, and if $|f(x) - \tilde{f}(x)| \geq 2\varepsilon$, then either $|f_j(x) - f(x)| \geq \varepsilon$ or else $|f_j(x) - \tilde{f}(x)| \geq \varepsilon$. It follows that

$$\{x \in X : |f(x) - \tilde{f}(x)| \geq 2\varepsilon\} \subset E_{j,\varepsilon} \cup \tilde{E}_{j,\varepsilon}$$

for all positive integers j, and therefore

$$\mu\left(\{x \in X : |f(x) - \tilde{f}(x)| \geq 2\varepsilon\}\right) \leq \mu(E_{j,\varepsilon}) + \mu(\tilde{E}_{j,\varepsilon})$$

for all positive integers j.

Taking limits as $j \to +\infty$, we conclude that

$$\mu\left(\{x \in X : |f(x) - \tilde{f}(x)| \geq 2\varepsilon\}\right) = 0$$

for all positive real numbers ε. Now

$$\{x \in X : f(x) \neq \tilde{f}(x)\} = \bigcup_{k=1}^{+\infty} \{x \in X : |f(x) - \tilde{f}(x)| \geq 2/k\}.$$

It follows that $\{x \in X : f(x) \neq \tilde{f}(x)\}$ is expressible as a countable union of subsets of X that are each of measure zero, and thus must itself be of measure zero. The result follows. $\blacksquare$

Proposition 9.6

Let $(X, \mathcal{A}, \mu)$ be a measure space, let $f_1, f_2, f_3, \ldots$ be an infinite sequence of measurable real-valued functions on X, and let f be a measurable real-valued function on X. Suppose that the infinite sequence $f_1, f_2, f_3, \ldots$ converges almost uniformly on X to the limit function f. Then the sequence $f_1, f_2, f_3, \ldots$ converges in measure on X to the limit function f.

Proof

Let strictly positive real numbers ε and δ be given. There then exists a measurable subset F of X such that $\mu(F) < \delta$ and $f_1, f_2, f_3, \ldots$ converges uniformly to f on $X \setminus F$. There then exists some positive integer N such that $|f_j(x) - f(x)| < \varepsilon$ whenever $x \in X \setminus F$ and $j \geq N$. But then

$$\{x \in X : |f_j(x) - f(x)| \geq \varepsilon\} \subset F$$

whenever $j \geq N$. But $\mu(F) < \delta$. It follows that

$$\mu\left(\{x \in X : |f_j(x) - f(x)| \geq \varepsilon\}\right) < \delta$$

whenever $j \geq N$. Thus $f_1, f_2, f_3, \ldots$ converges in measure to the limit function f, as required. ∎

9.4. Sequences of Functions that are Cauchy in Measure

Definition

Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $f_1, f_2, f_3, \ldots$ be an infinite sequence of measurable real-valued functions on X. The sequence $f_1, f_2, f_3, \ldots$ is said to to be *Cauchy in measure* if, given any positive real numbers ε and δ, there exists some positive integer N such that

$$\mu\left(\{x \in X : |f_j(x) - f_k(x)| \geq \varepsilon\}\right) < \delta$$

whenever $j \geq N$ and $k \geq N$.

Lemma 9.7

Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $f_1, f_2, f_3, \ldots$ be a sequence of measurable real-valued functions on X that converges in measure to some limit function. Then the infinite sequence $f_1, f_2, f_3, \ldots$ of functions is Cauchy in measure.

Proof

For all positive integers j and k and positive real numbers ε, let

$$E_{j,\varepsilon} = \{x \in X : |f_j(x) - f(x)| \geq \varepsilon\}$$

and

$$G_{j,k,\varepsilon} = \{x \in X : |f_j(x) - f_k(x)| \geq \varepsilon\}.$$

Let strictly positive real numbers ε and δ be given. Then there exists some positive integer N such that $\mu(E_{j,\frac{1}{2}\varepsilon}) < \frac{1}{2}\delta$ whenever $j \geq N$, because the sequence $f_1, f_2, f_3, \ldots$ converges in measure to the function f. Now if $x \in X$, if j and k are positive integers, and if $|f_j(x) - f_k(x)| \geq \varepsilon$ then either $|f_j(x) - f(x)| \geq \frac{1}{2}\varepsilon$ or else $|f_k(x) - f(x)| \geq \frac{1}{2}\varepsilon$. It follows that

$$G_{j,k,\varepsilon} \subset E_{j,\frac{1}{2}\varepsilon} \cup E_{k,\frac{1}{2}\varepsilon}$$

for all positive integers j and k, and therefore

$$\mu(G_{j,k,\varepsilon}) \leq \mu(E_{j,\frac{1}{2}\varepsilon}) + \mu(E_{k,\frac{1}{2}\varepsilon}) < \tfrac{1}{2}\delta + \tfrac{1}{2}\delta = \delta$$

whenever $j \geq N$ and $k \geq N$. The result follows. ∎

Proposition 9.8

Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $f_1, f_2, f_3, \ldots$ be a sequence of measurable real-valued functions on X that is Cauchy in measure. Then the sequence $f_1, f_2, f_3, \ldots$ has a subsequence that converges almost uniformly on X.

Proof

The sequence $f_1, f_2, f_3, \ldots$ is Cauchy in measure, and therefore there exists an infinite sequence $j_1, j_2, j_3, \ldots$ of positive integers, where

$$j_1 < j_2 < j_3 < \cdots ,$$

such that, for each positive integer k,

$$\mu\left(\left\{x \in X : |f_r(x) - f_s(x)| \geq \frac{1}{2^k}\right\}\right) < \frac{1}{2^k}$$

whenever $r \geq j_k$ and $s \geq j_k$. It then follows in particular that $\mu(E_k) < 2^{-k}$ for all positive integers k, where

$$E_k = \left\{x \in X : |f_{j_k}(x) - f_{j_{k+1}}(x)| \geq \frac{1}{2^k}\right\}.$$

Let $F_k = \bigcup_{p=k}^{+\infty} E_p$ for all positive integers k. Then

$$\mu(F_k) \leq \sum_{p=k}^{+\infty} \mu(E_p) \leq \sum_{p=k}^{+\infty} \frac{1}{2^p} = \frac{1}{2^{k-1}}.$$

for all positive integers k.

Let k be a positive integer, and let $x \in X \setminus F_k$. Then $x \notin E_p$ for all integers p satisfying $p \geq k$, and therefore $|f_{j_p}(x) - f_{j_{p+1}}(x)| < 2^{-p}$ for all $p \geq k$. Thus if p and q are integers satisfying $k \leq p < q$ then

$$|f_{j_p}(x) - f_{j_q}(x)| < \sum_{r=p}^{q-1} \frac{1}{2^r} < \frac{1}{2^{p-1}}.$$

It follows that, given any positive real number ε, there exists some positive integer M such that $|f_{j_p}(x) - f_{j_q}(x)| < \varepsilon$ for all integers p and q satisfying $M \leq p < q$. Thus $f_1(x), f_2(x), f_3(x), \ldots$ is a Cauchy sequence of real numbers. The completeness of the real number system ensures that every Cauchy sequence of real numbers is convergent. It follows that $f(x)$ is well-defined for all $x \in X \setminus F_k$, where $f(x) = \lim_{p \to +\infty} f_{j_p}(x)$.

Now we showed that $|f_{j_p}(x) - f_{j_q}(x)| < 2^{1-p}$ whenever $x \notin F_k$ and $k \leq p < q$. Taking limits as $q \to +\infty$, it follows that $|f_{j_p}(x) - f(x)| \leq 2^{1-p}$ whenever $p \geq k$. Therefore, given any positive real number ε, there exists a positive integer M_k, independent of the choice of x, such that $|f_{j_p}(x) - f(x)| < \varepsilon$ for all $x \in X \setminus F_k$ and for all integers p satisfying $p \geq M_k$. The sequence $f_{j_1}, f_{j_2}, f_{j_3}, \ldots$ of real-valued functions therefore converges uniformly on $X \setminus F_k$ to the limit function f.

Now $f(x)$ is well-defined for all positive integers k and for all points x of $X \setminus F_k$. It is therefore well-defined for all points x of $X \setminus F_\infty$, where $F_\infty = \bigcap_{k=1}^{+\infty} F_k$. Now F_∞ is a measurable set, and $\mu(F_\infty) \leq \mu(F_k) \leq 2^{1-k}$ for all positive integers. It follows that $\mu(F_\infty) = 0$. Thus the limit function f is defined almost everywhere on X. Also, given any positive real number δ, the positive integer k can be chosen large enough to ensure that $\mu(F_k) < \delta$. The sequence $f_{j_1}, f_{j_2}, f_{j_3}, \ldots$ then converges uniformly to the limit function f on $X \setminus F_k$. The sequence $f_{j_1}, f_{j_2}, f_{j_3}, \ldots$ is therefore almost uniformly convergent on X. Moreover it is a subsequence of the given sequence $f_1, f_2, f_3, \ldots$. The result follows. ∎

Proposition 9.9

Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $f_1, f_2, f_3, \ldots$ be a sequence of measurable real-valued functions on X that is Cauchy in measure. Then the sequence $f_1, f_2, f_3, \ldots$ converges in measure to some measurable real-valued function f on X.

Proof

The sequence $f_1, f_2, f_3, \ldots$ of functions is Cauchy in measure and therefore has a subsequence $f_{j_1}, f_{j_2}, f_{j_3}, \ldots$ that converges almost uniformly on X to some measurable real-valued function f (Proposition 9.8). We show that the sequence $f_1, f_2, f_3, \ldots$ converges in measure to this limit function f.

Let strictly positive real numbers ε and δ be given. The subsequence $f_{j_1}, f_{j_2}, f_{j_3}, \ldots$ converges in measure on X to the limit function f, because it converges almost uniformly to f on X (Proposition 9.6). Therefore there exists some positive integer M such that

$$\mu\left(\{x \in X : |f_{j_k}(x) - f(x)| \geq \tfrac{1}{2}\varepsilon\}\right) < \tfrac{1}{2}\delta$$

whenever $k \geq M$. Also there exists some positive integer N such that

$$\mu\left(\{x \in X : |f_p(x) - f_q(x)| \geq \tfrac{1}{2}\varepsilon\}\right) < \tfrac{1}{2}\delta$$

whenever $p \geq N$ and $q \geq N$, because the infinite sequence $f_1, f_2, f_3, \ldots$ is Cauchy in measure on X.

Choose a positive integer k large enough to ensure that $k \geq M$ and $j_k \geq N$, and let $g = f_{j_k}$. Also let

$$F_j = \{x \in X : |f_j(x) - g(x)| \geq \tfrac{1}{2}\varepsilon\}$$

for all positive integers j, and let

$$G = \{x \in X : |g(x) - f(x)| \geq \tfrac{1}{2}\varepsilon\}.$$

Then $\mu(F_j) < \frac{1}{2}\delta$ whenever $j \geq N$. Also $\mu(G) < \frac{1}{2}\delta$. Now if $x \in X$, and if $|f_j(x) - g(x)| < \frac{1}{2}\varepsilon$ and $|g(x) - f(x)| < \frac{1}{2}\varepsilon$ then

$$|f_j(x) - f(x)| \leq |f_j(x) - g(x)| + |g(x) - f(x)| < \varepsilon.$$

Thus

$$\{x \in X : |f_j(x) - f(x)| \geq \varepsilon\} \subset F_j \cup G$$

for all positive integers j.

It follows that if $j \geq N$ then

$$\mu\left(\{x \in X : |f_j(x) - f(x)| \geq \varepsilon\}\right) \leq \mu(F_j \cup G) \leq \mu(F_j) + \mu(G) < \delta.$$

Thus the infinite sequence $f_1, f_2, f_3, \ldots$ of measurable real-valued functions converges in measure to the limit function f, as required. ∎

9.5. Convergence in Mean

Definition

Let $(X, \mathcal{A}, \mu)$ be a measure space, let $f_1, f_2, f_3, \ldots$ be an infinite sequence of measurable real-valued functions on X, where $\int_X |f_j| \, d\mu < +\infty$ for all positive integers j, and let f be a measurable real-valued function on X. We say that the infinite sequence $f_1, f_2, f_3, \ldots$ of functions *converges in mean* to the function f if, given any strictly positive real number ε, there exists some positive integer N such that

$$\int_X |f_j - f| \, d\mu < \varepsilon$$

whenever $j \geq N$.

Definition

Let $(X, \mathcal{A}, \mu)$ be a measure space, let $f_1, f_2, f_3, \ldots$ be an infinite sequence of measurable real-valued functions on X, where $\int_X |f_j| \, d\mu < +\infty$ for all positive integers j. We say that the infinite sequence $f_1, f_2, f_3, \ldots$ of functions is *Cauchy in mean* if given any strictly positive real number ε, there exists some positive integer N such that

$$\int_X |f_j - f_k| \, d\mu < \varepsilon$$

whenever $j \geq N$ and $k \geq N$.

Proposition 9.10

Let $(X, \mathcal{A}, \mu)$ be a measure space, let $f_1, f_2, f_3, \ldots$ be an infinite sequence of measurable real-valued functions on X, where $\int_X |f_j| \, d\mu < +\infty$ for all positive integers j, and let f be a measurable real-valued function on X. Suppose that the infinite sequence $f_1, f_2, f_3, \ldots$ of functions on X is Cauchy in mean and also converges almost everywhere on X to the limit function f. Then the infinite sequence $f_1, f_2, f_3, \ldots$ of functions on X converges in mean to the function f.

Proof

Let some strictly positive real number ε be given, and let ε_0 satisfy $0 < \varepsilon_0 < \varepsilon$. The infinite sequence $f_1, f_2, f_3, \ldots$ of functions is Cauchy in mean, hence there exists some positive integer N such that $\int_X |f_j - f_k| \, d\mu < \varepsilon_0$ whenever $j \geq N$ and $k \geq N$. Taking limits as $k \to +\infty$, and applying Fatou's Lemma (Lemma 8.25), we find that

$$\begin{aligned} \int_X |f_j - f| \, d\mu &= \int_X \left(\lim_{k \to +\infty} |f_j - f_k| \right) d\mu \\ &\leq \liminf_{k \to +\infty} \int_X |f_j - f_k| \, d\mu \leq \varepsilon_0 < \varepsilon \end{aligned}$$

whenever $j \geq N$, Therefore $\lim\limits_{j \to +\infty} \int_X |f_j - f| \, d\mu = 0$, and thus f_j converges to f in mean, as required. ∎

Lemma 9.11

Let $(X, \mathcal{A}, \mu)$ be a measure space, let $f_1, f_2, f_3, \ldots$ be an infinite sequence of measurable real-valued functions on X, where $\int_X |f_j| \, d\mu < +\infty$ for all positive integers j, and let f be a measurable real-valued function on X. Suppose that the infinite sequence $f_1, f_2, f_3, \ldots$ of functions on X is converges in mean to the limit function f. Then this infinite sequence of functions also converges in measure to the function f.

Proof

Let

$$E_{j,\varepsilon} = \{x \in X : |f_j(x) - f(x)| \geq \varepsilon\}$$

and let $\chi_{j,\varepsilon}$ denote the characteristic function of the set $E_{j,\varepsilon}$ for all positive integers j and for all positive real numbers ε. Then $|f_j(x) - f(x)| \geq \varepsilon \chi_{j,\varepsilon}(x)$ for all $x \in X$, positive integers j and positive real numbers ε, and therefore

$$\int_X |f_j - f| \, dx \geq \varepsilon \mu(E_{j,\varepsilon})$$

for all positive integers j and for all positive real number ε.

Now let positive real numbers ε and δ be given. The sequence $f_1, f_2, f_3, \ldots$ is Cauchy in mean. It follows that there exists some positive integer N such that $\int_X |f_j - f| \, d\mu < \varepsilon\delta$ whenever $j \geq N$. Then $\mu(E_{j,\varepsilon}) < \delta$ whenever $j \geq N$. The result follows. ∎

Lemma 9.12

Let $(X, \mathcal{A}, \mu)$ be a measure space, let $f_1, f_2, f_3, \ldots$ be an infinite sequence of measurable real-valued functions on X, where $\int_X |f_j| \, d\mu < +\infty$ for all positive integers j. Suppose that the infinite sequence $f_1, f_2, f_3, \ldots$ of functions on X is Cauchy in mean. Then this infinite sequence of functions is also Cauchy in measure.

Proof

Let

$$E_{j,k,\varepsilon} = \{x \in X : |f_j(x) - f_k(x)| \geq \varepsilon\}$$

and let $\chi_{j,k,\varepsilon}$ denote the characteristic function of the set $E_{j,k,\varepsilon}$ for all positive integers j and k and for all positive real numbers ε. Then $|f_j(x) - f_k(x)| \geq \varepsilon \chi_{j,k,\varepsilon}(x)$ for all $x \in X$, positive integers j and k and positive real numbers ε, and therefore

$$\int_X |f_j - f_k| \, dx \geq \varepsilon \mu(E_{j,k,\varepsilon})$$

for all positive integers j and k and for all positive real number ε.

Now let positive real numbers ε and δ be given. The sequence $f_1, f_2, f_3, \ldots$ is Cauchy in mean. It follows that there exists some positive integer N such that $\int_X |f_j - f_k| \, d\mu < \varepsilon\delta$ whenever $j \geq N$ and $k \geq N$. Then $\mu(E_{j,k,\varepsilon}) < \delta$ whenever $j \geq N$ and $k \geq N$. The result follows. ∎

Proposition 9.13

Let $(X, \mathcal{A}, \mu)$ be a measure space, let $f_1, f_2, f_3, \ldots$ be an infinite sequence of measurable real-valued functions on X, where $\int_X |f_j| \, d\mu < +\infty$ for all positive integers j. Suppose that the infinite sequence $f_1, f_2, f_3, \ldots$ of functions on X is Cauchy in mean. Then this infinite sequence of functions converges in mean to some measurable real-valued function f for which $\int_X |f| \, d\mu < +\infty$.

Proof

The infinite sequence $f_1, f_2, f_3, \ldots$ of functions is Cauchy in mean. It is therefore Cauchy in measure (Lemma 9.12). It therefore has a subsequence $f_{j_1}, f_{j_2}, f_{j_3}, \ldots$ that converges almost uniformly on X to some measurable real-valued function f on X (Proposition 9.8). This subsequence converges pointwise almost everywhere on X to the limit function f, (Lemma 9.3), and therefore converges in mean to the function f (Proposition 9.10). A positive integer k can then be chosen large enough to ensure that $\int_X |f_{j_k} - f| \, d\mu \leq 1$. It then follows that

$$\int_X |f| \, d\mu \leq \int_X |f_{j_k}| \, d\mu + \int_X |f_{j_k} - f| \, d\mu \leq \int_X |f_{j_k}| \, d\mu + 1 < +\infty.$$

To complete the proof we show that the original sequence $f_1, f_2, f_3, \ldots$ converges in mean to the limit function f.

Let some strictly positive real number ε be given. Then there exist positive integers M and N that are large enough to ensure that $\int_X |f_{j_k} - f| \, d\mu < \frac{1}{2}\varepsilon$ whenever $k \geq M$ and $\int_X |f_s - f_t| \, d\mu < \frac{1}{2}\varepsilon$ whenever $s \geq N$ and $t \geq N$. Let some positive integer k be chosen large enough to ensure that $k \geq M$ and $j_k \geq N$. Then

$$\int_X |f_n - f| \, d\mu \leq \int_X |f_n - f_{j_k}| \, d\mu + \int_X |f_{j_k} - f| \, d\mu < \varepsilon$$

whenever $n \geq N$. Thus $\lim\limits_{n \to +\infty} \int_X |f_n - f| \, d\mu = 0$. The result follows. ∎

9.6. Hölder's Inequality

Lemma 9.14 (Young's Inequality)

Let a and b be non-negative real numbers, and let p and q be real numbers for which $p > 1$, $q >$ and $\dfrac{1}{p} + \dfrac{1}{q} = 1$. Then

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

Proof

First we show that $e^{(1-t)u+tv} \leq (1-t)e^u + te^v$ for all real numbers u, v and t satisfying $0 \leq t \leq 1$. This is a consequence of the fact that the derivative of the exponential function is increasing. The result clearly holds when $u = v$, and moreover the form of the inequality is preserved on swapping u and v and replacing t by $1 - t$. It therefore suffices to verify that the above inequality holds in the case when $u < v$.

Suppose then that $u < v$, and let

$$f(t) = (1-t)e^u + te^v - e^{(1-t)u+tv}.$$

for all real numbers t. Then $f(0) = f(1) = 0$ and

$$f'(t) = e^v - e^u - (v-u)e^{(1-t)u+tv}.$$

From this expression we see that $f'(t)$ is a decreasing function of t. It must therefore be the case that $f'(0) > 0$ and $f'(1) < 0$. This is then a real number t_0 satisfying $0 < t_0 < 1$ for which $f'(t_0) = 0$. The function f is then an increasing function of t on the interval $[0, t_0]$ and a decreasing function of t on the interval $[t_0, 1]$. But $f(0) = f(1) = 0$. It follows that $f(t) > 0$ whenever $0 < t < 1$, and thus $e^{(1-t)u+tv} < (1-t)e^u + te^v$ for all real numbers u, v and t satisfying $u < v$ and $0 < t < 1$. This completes the verification that $e^{(1-t)u+tv} \leq (1-t)e^u + te^v$ in all cases where u, v and t are real numbers and $0 \leq t \leq 1$.

Now let a and b be positive real numbers, and let p and q be real numbers for which $p > 1$, $q > 1$ and $p^{-1} + q^{-1} = 1$. Let $u = p \log a$, $v = q \log b$ and $t = 1/q$. Then $1 - t = 1/p$, and therefore

$$e^{(1-t)u+tv} = e^{\log a + \log b} = e^{\log a} e^{\log b} = ab.$$

Also

$$(1-t)e^u = \frac{1}{p} e^{p \log a} = \frac{a^p}{p} \quad \text{and} \quad te^v = \frac{1}{q} e^{q \log b} = \frac{b^q}{q}.$$

The inequality previously established therefore ensures that

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

for all positive real numbers a and b. This inequality also holds when $a \geq 0$, $b \geq 0$ and either $a = 0$ or $b = 0$. The result follows. $\blacksquare$

Alternative Proof

Let p and q be real numbers satisfying $p > 1$, $q > 1$ and $p^{-1} + q^{-1} = 1$. Then $q - 1 = 1/(p - 1)$. It follows that if x and y are positive real numbers then $y = x^{p-1}$ if and only if $x = y^{q-1}$. Let a and b be positive real numbers. Then the rectangle in $\mathbb{R}^2$ with vertices $(0,0)$, $(a,0)$, (a,b) and $(0,b)$ is contained in the union of the regions A and B, where

$$\begin{aligned} A &= \{(x,y) \in \mathbb{R}^2 : 0 \leq x \leq a \text{ and } 0 \leq y \leq x^{p-1}\}, \\ B &= \{(x,y) \in \mathbb{R}^2 : 0 \leq y \leq b \text{ and } 0 \leq x \leq y^{q-1}\}. \end{aligned}$$

It follows that

$$ab \leq \int_0^a x^{p-1}\,dx + \int_0^b y^{q-1}\,dy = \frac{a^p}{p} + \frac{b^q}{q},$$

as required. ∎

Proposition 9.15 (Hölder's Inequality)

Let $(X, \mathcal{A}, \mu)$ be a measure space, let p and q be real numbers satisfying the conditions $p > 1$, $q > 1$ and $p^{-1} + q^{-1} = 1$, and let f and g be measurable real-valued functions on X. Suppose that the functions $|f|^p$ and $|g|^q$ are integrable. Then the function fg is integrable and

$$\int_X |fg|\,d\mu \leq \left(\int_X |f|^p\,d\mu\right)^{\frac{1}{p}} \left(\int_X |g|^q\,d\mu\right)^{\frac{1}{q}}.$$

Proof

Let

$$\|f\|_p = \left(\int_X |f|^p \, d\mu\right)^{\frac{1}{p}} \quad \text{and} \quad \|g\|_q = \left(\int_X |g|^q \, d\mu\right)^{\frac{1}{q}}.$$

The identity $ab \leq p^{-1}a^p + q^{-1}b^q$ holds for all non-negative real numbers a and b. (Lemma 9.14). Setting $a = |f(x)|/\|f\|_p$ and $b = |g(x)|/\|g\|_q$, and integrating over the space X we find that

$$\begin{aligned} \frac{1}{\|f\|_p \, \|g\|_q} \int_X |fg| \, d\mu &\leq \frac{1}{p\,\|f\|_p^p} \int_X |f|^p \, d\mu + \frac{1}{q\,\|g\|_q^q} \int_X |g|^q \, d\mu \\ &= \frac{1}{p} + \frac{1}{q} = 1. \end{aligned}$$

Thus $\int_X |fg| \, d\mu \leq \|f\|_p \, \|g\|_q$, as required. ∎

9.7. Minkowski's Inequality

Proposition 9.16 (Minkowski's Inequality)

Let $(X, \mathcal{A}, \mu)$ be a measure space, let p be a real number satisfying the conditions $p \geq 1$, and let f and g be measurable real-valued functions on X. Suppose that the functions $|f|^p$ and $|g|^p$ are integrable. Then the function $|f+g|^p$ is integrable and

$$\left(\int_X |f+g|^p \, d\mu \right)^{\frac{1}{p}} \leq \left(\int_X |f|^p \, d\mu \right)^{\frac{1}{p}} + \left(\int_X |g|^p \, d\mu \right)^{\frac{1}{p}}.$$

Proof

The result in the case $p = 1$ follows immediately on integrating the inequality $|f + g| \leq |f| + |g|$ over the measure space X. It remains therefore to prove the result in the case $p > 1$. There then exists some positive real number q, such that $q > 1$ and $p^{-1} + q^{-1} = 1$. Then $1/q = (p - 1)/p$, and thus $p = (p - 1)q$. Let

$$\|f\|_p = \left(\int_X |f|^p \, d\mu \right)^{\frac{1}{p}}, \quad \|g\|_p = \left(\int_X |g|^p \, d\mu \right)^{\frac{1}{p}}$$

and

$$\|f + g\|_p = \left(\int_X |f + g|^p \, d\mu \right)^{\frac{1}{p}}.$$

We may assume moreover that $\|f + g\|_p > 0$, since the required inequality is trivially satisfied in cases where $\|f + g\| = 0$. Now, applying Hölder's Inequality (Proposition 9.15), we find that

$$\begin{aligned}
\|f+g\|_p^p &= \int_X |f+g|^p \, d\mu \\
&\leq \int_X |f+g|^{p-1}(|f|+|g|) \, d\mu \\
&\leq \left(\int_X |f+g|^{(p-1)q} \, d\mu \right)^{\frac{1}{q}} (\|f\|_p + \|g\|_p) \\
&= \left(\int_X |f+g|^p \, d\mu \right)^{1-\frac{1}{p}} (\|f\|_p + \|g\|_p) \\
&= \|f+g\|_p^{p-1} (\|f\|_p + \|g\|_p)
\end{aligned}$$

Thus $\|f+g\|_p^p \leq \|f+g\|_p^{p-1} (\|f\|_p + \|g\|_p)$. Dividing both sides of the inequality by $|f+g|_p^{p-1}$, we find that $\|f+g\|_p \leq \|f\|_p + \|g\|_p$, as required. ∎

9.8. Convergence in L^p norm

Definition

Let $(X, \mathcal{A}, \mu)$ be a measure space, let p be a real number satisfying $p \geq 1$, let $f_1, f_2, f_3, \ldots$ be an infinite sequence of measurable real-valued functions on X, where $\int_X |f_j|^p \, d\mu < +\infty$ for all positive integers j, and let f be a measurable real-valued function on X. We say that the infinite sequence $f_1, f_2, f_3, \ldots$ of functions *converges in L^p norm* to the function f if, given any strictly positive real number ε, there exists some positive integer N such that

$$\int_X |f_j - f|^p \, d\mu < \varepsilon^p$$

whenever $j \geq N$.

Given a measure space $(X, \mathcal{A}, \mu)$, and given a real number p satisfying $p \geq 1$, it is convenient to define

$$\|f\|_p = \left(\int_X |f|^p \, d\mu \right)^{\frac{1}{p}}$$

for all measurable real-valued functions f on X for which $\int_X |f|^p \, d\mu < +\infty$. With this notation, we can say that if $f_1, f_2, f_3, \ldots$ is an infinite sequence of measurable real-valued functions on X, where $\int_X |f|^p \, d\mu < +\infty$, and if f is a measurable real-valued function on X, then the infinite sequence f_1, f_2, f_3 of functions converges to the the limit function f in L^p norm if and only if $\lim_{j \to +\infty} \|f_j - f\|_p = 0$.

Definition

Let $(X, \mathcal{A}, \mu)$ be a measure space, let p be a real number satisfying $p \geq 1$, let $f_1, f_2, f_3, \ldots$ be an infinite sequence of measurable real-valued functions on X, where $\int_X |f_j|^p \, d\mu < +\infty$ for all positive integers j. We say that the infinite sequence $f_1, f_2, f_3, \ldots$ of functions is *Cauchy in L^p norm* if given any strictly positive real number ε, there exists some positive integer N such that

$$\int_X |f_j - f_k|^p \, d\mu < \varepsilon^p$$

whenever $j \geq N$ and $k \geq N$.

Let $(X, \mathcal{A}, \mu)$ be a measure space, let p be a real number satisfying $p \geq 1$, let $f_1, f_2, f_3, \ldots$ be an infinite sequence of measurable real-valued functions on X, where $\int_X |f_j|^p \, d\mu < +\infty$ for all positive integers j. The infinite sequence $f_1, f_2, f_3, \ldots$ of functions on X is then *Cauchy in L^p norm* if, given any positive real number ε, there exists some positive integer N such that $\|f_j - f_k\|_p < \varepsilon$ whenever $j \geq N$ and $k \geq N$.

An infinite sequence of measurable real-valued functions on a measure space converges in L^1 norm if and only if it converges in mean. Similarly an infinite sequence of measurable real-valued functions on a measure space is Cauchy in L^1 norm if and only if it is Cauchy in mean. The following results, and their proofs, accordingly generalize those previously stated and proved for sequences of functions that converge in mean or are Cauchy in mean.

Proposition 9.17

Let $(X, \mathcal{A}, \mu)$ be a measure space, let p be a real number satisfying $p \geq 1$, let $f_1, f_2, f_3, \ldots$ be an infinite sequence of measurable real-valued functions on X, where $\int_X |f_j|^p \, d\mu < +\infty$ for all positive integers j, and let f be a measurable real-valued function on X. Suppose that the infinite sequence $f_1, f_2, f_3, \ldots$ of functions on X is Cauchy in L^p norm and also converges almost everywhere on X to the limit function f. Then the infinite sequence $f_1, f_2, f_3, \ldots$ of functions on X converges in L^p norm to the function f.

Proof

Let some strictly positive real number ε be given, and let ε_0 satisfy $0 < \varepsilon_0 < \varepsilon$. The infinite sequence $f_1, f_2, f_3, \ldots$ of functions is Cauchy in L^p norm, hence there exists some positive integer N such that $\|f_j - f_k\|_p < \varepsilon_0$ whenever $j \geq N$ and $k \geq N$. Taking limits as $k \to +\infty$, and applying Fatou's Lemma (Lemma 8.25), we find that

$$\begin{aligned} \int_X |f_j - f|^p \, d\mu &= \int_X \left(\lim_{k \to +\infty} |f_j - f_k|^p \right) d\mu \\ &\leq \liminf_{k \to +\infty} \int_X |f_j - f_k|^p \, d\mu \leq \varepsilon_0^p < \varepsilon^p \end{aligned}$$

whenever $j \geq N$, Therefore $\lim_{j \to +\infty} \|f_j - f\|_p = 0$, and thus f_j converges to f in L^p norm, as required. ∎

Lemma 9.18

Let $(X, \mathcal{A}, \mu)$ be a measure space, let p be a real number satisfying $p \geq 1$, let $f_1, f_2, f_3, \ldots$ be an infinite sequence of measurable real-valued functions on X, where $\int_X |f_j|^p \, d\mu < +\infty$ for all positive integers j, and let f be a measurable real-valued function on X. Suppose that the infinite sequence $f_1, f_2, f_3, \ldots$ of functions on X is converges in L^p norm to the limit function f. Then this infinite sequence of functions also converges in measure to the function f.

Proof

Let

$$E_{j,\varepsilon} = \{x \in X : |f_j(x) - f(x)| \geq \varepsilon\}$$

and let $\chi_{j,\varepsilon}$ denote the characteristic function of the set $E_{j,\varepsilon}$ for all positive integers j and for all positive real numbers ε. Then $|f_j(x) - f(x)|^p \geq \varepsilon^p \chi_{j,\varepsilon}(x)$ for all $x \in X$, positive integers j and positive real numbers ε, and therefore

$$\int_X |f_j - f|^p \, dx \geq \varepsilon^p \mu(E_{j,\varepsilon})$$

for all positive integers j and for all positive real number ε.

Now let positive real numbers ε and δ be given. The sequence $f_1, f_2, f_3, \ldots$ is Cauchy in L^p norm. It follows that there exists some positive integer N such that $\|f_j - f\|_p < \varepsilon\delta^{\frac{1}{p}}$ whenever $j \geq N$. Then $\mu(E_{j,\varepsilon}) < \delta$ whenever $j \geq N$. The result follows. $\blacksquare$

Lemma 9.19

Let $(X, \mathcal{A}, \mu)$ be a measure space, let p be a real number satisfying $p \geq 1$, let $f_1, f_2, f_3, \ldots$ be an infinite sequence of measurable real-valued functions on X, where $\int_X |f_j|^p \, d\mu < +\infty$ for all positive integers j. Suppose that the infinite sequence $f_1, f_2, f_3, \ldots$ of functions on X is Cauchy in L^p norm. Then this infinite sequence of functions is also Cauchy in measure.

Proof

Let

$$E_{j,k,\varepsilon} = \{x \in X : |f_j(x) - f_k(x)| \geq \varepsilon\}$$

and let $\chi_{j,k,\varepsilon}$ denote the characteristic function of the set $E_{j,k,\varepsilon}$ for all positive integers j and k and for all positive real numbers ε. Then $|f_j(x) - f_k(x)|^p \geq \varepsilon^p \chi_{j,k,\varepsilon}(x)$ for all $x \in X$, positive integers j and k and positive real numbers ε, and therefore

$$\int_X |f_j - f_k|^p \, dx \geq \varepsilon^p \mu(E_{j,k,\varepsilon})$$

for all positive integers j and k and for all positive real number ε.

Now let positive real numbers ε and δ be given. The sequence $f_1, f_2, f_3, \ldots$ is Cauchy in L^p norm. It follows that there exists some positive integer N such that $\|f_j - f_k\|_p < \varepsilon\delta^{\frac{1}{p}}$ whenever $j \geq N$ and $k \geq N$. Then $\mu(E_{j,k,\varepsilon}) < \delta$ whenever $j \geq N$ and $k \geq N$. The result follows. ∎

Proposition 9.20

Let $(X, \mathcal{A}, \mu)$ be a measure space, let p be a real number satisfying $p \geq 1$, let $f_1, f_2, f_3, \ldots$ be an infinite sequence of measurable real-valued functions on X, where $\int_X |f_j|^p \, d\mu < +\infty$ for all positive integers j. Suppose that the infinite sequence $f_1, f_2, f_3, \ldots$ of functions on X is Cauchy in L^p norm. Then this infinite sequence of functions converges in L^p norm to some measurable real-valued function f for which $\int_X |f|^p \, d\mu < +\infty$.

Proof

The infinite sequence $f_1, f_2, f_3, \ldots$ of functions is Cauchy in L^p norm. It is therefore Cauchy in measure (Lemma 9.19). It therefore has a subsequence $f_{j_1}, f_{j_2}, f_{j_3}, \ldots$ that converges almost uniformly on X to some measurable real-valued function f on X (Proposition 9.8). The subsequence $f_{j_1}, f_{j_2}, f_{j_3}, \ldots$ then converges pointwise almost everywhere on X to the limit function f (Lemma 9.3), and therefore converges in L^p norm to the function f (Proposition 9.17). A positive integer k can then be chosen large enough to ensure that $\|f_{j_k} - f\|_p \leq 1$. It then follows from Minkowski's inequality (Proposition 9.16) that

$$\left(\int_X |f|^p \, d\mu\right)^{\frac{1}{p}} \leq \|f_{j_k}\|_p + \|f_{j_k} - f\|_p \leq \|f_{j_k}\|_p + 1 < +\infty.$$

To complete the proof we show that the original sequence $f_1, f_2, f_3, \ldots$ converges in L^p norm to the limit function f.

Let some strictly positive real number ε be given. Then there exist positive integers M and N that are large enough to ensure that $\|f_{j_k} - f\|_p < \frac{1}{2}\varepsilon$ whenever $k \geq M$ and $\|f_s - f_t\|_p < \frac{1}{2}\varepsilon$ whenever $s \geq N$ and $t \geq N$. Let some positive integer k be chosen large enough to ensure that $k \geq M$ and $j_k \geq N$. Then

$$\|f_n - f\|_p \leq \|f_n - f_{j_k}\|_p + \|f_{j_k} - f\|_p < \varepsilon$$

whenever $n \geq N$. Thus $\lim\limits_{n \to +\infty} \|f_n - f\|_p = 0$. The result follows. ∎

9.9. The L^p Spaces

Let $(X, \mathcal{A}, \mu)$ be a measure space, let p be a real number satisfying $p \geq 1$, and let $\mathcal{L}^p(X, \mu)$ denote the collection consisting of all measurable real-valued functions f on X with the property that $\int_X |f|^p \, d\mu < +\infty$. It follows from Minkowski's Inequality (Proposition 9.16) that the sum of two real-valued functions belonging to $\mathcal{L}^p(X, \mu)$ itself belongs to $\mathcal{L}^p(X, \mu)$, and thus $\mathcal{L}^p(X, \mu)$ is a real vector space.

Let

$$\|f\|_p = \left(\int_X |f|^p \, d\mu \right)^{\frac{1}{p}}$$

for all $f \in \mathcal{L}^p(X, \mu)$. Then $\|f\|_p \geq 0$ and $\|cf\|_p = |c| \, \|f\|_p$ for all $f \in \mathcal{L}^p(X, \mu)$ and for all real numbers c. Also Minkowski's Inequality (Proposition 9.16) ensures that $\|f + g\|_p \leq \|f\|_p + \|g\|_p$ for all $f, g \in \mathcal{L}^p(X, \mu)$.

However it is not the case that $\|f\|_p = 0$ implies that $f = 0$. In fact $\|f\|_p = 0$ if and only if the set $\{x \in X : f(x) \neq 0\}$ has measure zero, so that the function f is equal to zero almost everywhere on X.

Now it is an easy exercise to verify that the relation of being equal almost everywhere is an equivalence relation on the set of functions that belong to $\mathcal{L}^p(X, \mu)$. This relation therefore partitions the set $\mathcal{L}^p(X, \mu)$ into equivalence claases. Let f be a measurable real-valued function on X for which $|f|^p$ is integrable on X. The equivalence class $[f]_p$ of f then consists of those measurable real-valued functions on X that are equal to the function f almost everywhere on X.

Now if f, g, $\tilde{f}$ and $\tilde{g}$ are measurable functions on X, if c is a real numher, if f and $\tilde{f}$ are equal almost everywhere on X, and if g and $\tilde{g}$ are equal almost everywhere on X, then cf and $c\tilde{f}$ are equal almost everywhere on X, and $f + g$ and $\tilde{f} + \tilde{g}$ are equal almost everywhere on X. We may therefore add together equivalence classes of functions belonging to $\mathcal{L}^p(X, \mu)$ and multiply them by real scalars so that $[f]_p + [g]_p = [f + g]_p$ and $[cf]_p = c[f_p]$ for all $f, g \in \mathcal{L}^p(X, \mu)$ and for all real numbers c. Also $\|f\|_p = \|\tilde{f}\|_p$ whenever the measurable real-valued functions f and $\tilde{f}$ are equal almost everywehre on X and belong to the vector space $\mathcal{L}^p(X, \mu)$. We may therefore define $\|[f]_p\|_p = \|f\|_p$ for all $f \in \mathcal{L}^p(X, \mu)$.

Now let $L^p(X, \mu)$ denote the set of all equivalence classes $[f]_p$ of measurable real-valued functions on X that belong to $\mathcal{L}^p(X, \mu)$, where two such functions on X belong to the same equivalence class if and only if they are equal almost everywhere on X. Denoting the equivalence class of any member f of $\mathcal{L}^p(X, \mu)$ by $[f]_p$, we note that there are well-defined operations of addition and multiplication-by-scalars on $L^p(X, \mu)$, where

$$[f]_p + [g]_p = [f + g]_p, \quad c[f]_p = [cf]_p$$

for all $f, g \in \mathcal{L}^p(X, \mu)$ and for all real numbers c.

Also setting $\|[f]_p\|_p = \|f\|_p$ for all $f \in \mathcal{L}^p(X,\mu)$, where

$$\|f\|_p = \left(\int_X |f|^p \, d\mu\right)^{\frac{1}{p}},$$

we find that $\|[f]_p\|_p \geq 0$, $\|c[f]_p\|_p = |c| \, \|[f]_p\|_p$ and $\|[f]_p + [g]_p\|_p \leq \|[f]_p\|_p + \|[g]_p\|_p$ for all $f, g \in \mathcal{L}(X,\mu)$ and for all real numbers c. Also $\|[f]_p\|_p = 0$ if and only if $[f]_p = [0]_p$. It follows that the function that maps the equivalence class $[f]_p$ of each member f of $\mathcal{L}^p(X,\mu)$ to the non-negative real number $\|f\|_p$ is a norm on the vector space $L^p(X,\mu)$. We obtain in this fashion a normed vector space $L^p(X,\mu)$ whose elements are equivalence classes of measurable real-valued functions f on X. A measurable real-valued function f on X determines a corresponding element of $L^p(X,\mu)$ if and only if $\int_X |f|^p \, d\mu < +\infty$. Two such functions determine the same element of $L^p(X,\mu)$ if and only if they are equal almost everywhere on X with respect to the measure μ.

Proposition 9.20 ensures that the normed vector space $L^p(X, \mu)$ is complete for all real numbers p satisfying $p \geq 1$. The space $L^p(X, \mu)$ is thus a *Banach space*. (Banach spaces are complete normed vector spaces.) This result is of fundamental importance in many branches of mathematics developed from the early twentieth century onwards.