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Section 4: Countable and Uncountable Sets

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4. Countable and Uncountable Sets

4.1. Functions between Sets

Let X and Y be sets, and let $f\colon X \to Y$ be a function from X to Y. The function f is *injective* if, given any element y of Y, there exists at most one element x of X for which $f(x) = y$. The function f is *surjective* if, given any element y of Y, there exists at least one element x of X for which $f(x) = y$. The function f is *bijective* if it is both injective and surjective. Thus the function $f\colon X \to Y$ is bijective if and only if, given any element y of Y, there exists a exactly one element x of X for which $f(x) = y$. A function $f\colon X \to Y$ is bijective if and only if it has a well-defined inverse $f^{-1}\colon Y \to X$. Injective, surjective and bijective functions may be referred to as *injections*, *surjections* and *bijections* respectively.

4.2. Countable Sets

Definition

A non-empty set X is said to be *countable* if there exists an injection mapping X into the set $\mathbb{N}$ of positive integers. The empty set $\emptyset$ is also said to be countable.

Lemma 4.1

Any subset of a countable set is countable.

Proof

Let Y be a subset of a countable set X. Then there exists an injection $f\colon X \to \mathbb{N}$ from X to the set $\mathbb{N}$ of positive integers. The restriction of this injection to the set Y gives an injection from Y to $\mathbb{N}$. $\blacksquare$

Lemma 4.2

Let X and Y be sets, and let $f: X \to Y$ be an injective function from X to Y. Suppose that the set Y is countable. Then the set X is countable.

Proof

The set Y is countable, and therefore there exists an injective function $g: Y \to \mathbb{N}$ mapping Y into the set $\mathbb{N}$ of positive integers. Then the composition function $g \circ f: X \to \mathbb{N}$ is injective, because the composition of any two injective functions is always itself an injective function. It follows that the set X is countable, as required. ∎

We establish a one-to-one correspondence between the set $\mathbb{N} \times \mathbb{N}$ of ordered pairs of positive integers and the set $\mathbb{N}$ of positive integers. This correspondence is implemented by a function $h\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is constructed so that

$$h(1,1) = 1,$$

$$h(2,1) = 2, \quad h(1,2) = 3,$$

$$h(3,1) = 4, \quad h(2,2) = 5, \quad h(1,3) = 6,$$

$$h(4,1) = 7, \quad h(3,2) = 8, \quad h(2,3) = 9, \quad h(1,4) = 10, \quad \text{etc.}$$

The expression for the function h should be determined so that $h(j,k) = S(j+k) + k$ for all positive integers j and k, where, for each integer m satisfying $m \geq 2$, $S(m)$ is equal to the number of ordered pairs (p,q) of positive integers satisfying $p + q < m$.

Let m be a positive integer satisfying $m \geq 3$. Then, for each integer p between 1 and $m-2$, there are $m-p-1$ positive integers q satisfying $p+q<m$. It follows that

$$S(m) = \sum_{p=1}^{m-2} (m-p-1) = \sum_{i=1}^{m-2} i = \tfrac{1}{2}(m-1)(m-2).$$

This identity also holds when $m=2$, since $S(2)=0$. The function $h\colon \mathbb{N}\times\mathbb{N}\to\mathbb{N}$ constructed to implement the one-to-one correspondence between the sets $\mathbb{N}\times\mathbb{N}$ and $\mathbb{N}$ therefore satisfies

$$h(j,k) = \tfrac{1}{2}(j+k-1)(j+k-2)+k$$

for all positive integers j and k. We now prove formally that this function is indeed a bijection between the sets $\mathbb{N}\times\mathbb{N}$ and $\mathbb{N}$.

Lemma 4.3

Let $h\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be the function defined such that

$$h(j,k) = \tfrac{1}{2}(j+k-1)(j+k-2) + k.$$

for all positive integers j and k. Then $h\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is a bijection.

Proof

Let n be a positive integer. Then there is a unique integer m satisfying $m \geq 2$ for which

$$\tfrac{1}{2}(m-1)(m-2) < n \leq \tfrac{1}{2}m(m-1).$$

Let $k = n - \frac{1}{2}(m-1)(m-2)$ and $j = m - k$. Then j and k are integers between 1 and $m-1$, and

$$h(j,k) = \tfrac{1}{2}(m-1)(m-2) + k = n.$$

Now let j' and k' be positive integers satisfying $h(j', k') = n$. Then

$$0 < n - \tfrac{1}{2}(j' + k' - 1)(j' + k' - 2) = k' \leq j' + k' - 1,$$

and therefore

$$\tfrac{1}{2}(j' + k' - 1)(j' + k' - 2) < n \leq \tfrac{1}{2}(j' + k')(j' + k' - 1).$$

It follows that $j' + k' = m$, where m is the unique integer satisfying $m \geq 2$ for which

$$\tfrac{1}{2}(m - 1)(m - 2) < n \leq \tfrac{1}{2}m(m - 1).$$

But then

$$\tfrac{1}{2}(m - 1)(m - 2) + k' = n = \tfrac{1}{2}(m - 1)(m - 2) + k,$$

and therefore $k' = k$ and $j' = j$. Thus (j, k) is the unique ordered pair of positive integers for which $h(j, k) = n$. We have thus shown that, given any positive integer n, there exists a unique ordered pair (j, k) of positive integers for which $h(j, k) = n$. It follows that $h\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is a bijection, as required. $\blacksquare$

Lemma 4.4

Let $h\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ *be the function defined such that*

$$h(j,k) = \tfrac{1}{2}(j+k-1)(j+k-2) + k.$$

for all positive integers j *and* k, *and let functions*

$$g_n\colon \mathbb{N}^n \to \mathbb{N}$$

be constructed for $n = 1, 2, 3\ldots$ *so that* $g_1(j) = j$ *for all positive integers* j *and*

$$g_n(j_1, j_2, \ldots, j_n) = h(g_{n-1}(j_1, j_2, \ldots, j_{n-1}), j_n)$$

for all $(j_1, j_2, \ldots, j_n) \in \mathbb{N}^n$ *whenever* $n > 1$. *Then each of the functions* $g_n\colon \mathbb{N}^n \to \mathbb{N}$ *is a bijection.*

Proof

The function $g_1 \colon \mathbb{N} \to \mathbb{N}$ is a bijection because it is the identity function of $\mathbb{N}$. The function $g_2 \colon \mathbb{N}^2 \to \mathbb{N}$ coincides with the function h. It therefore follows from Lemm 4.3 that the function $g_2 \colon \mathbb{N}^2 \to \mathbb{N}$ is a bijection. We prove by induction on n that the function $g_n \colon \mathbb{N}^n \to \mathbb{N}$ is a bijection for all positive integers n. Suppose therefore as our induction hypothesis that n is some positive integer satisfying $n \geq 3$ and that $g_{n-1} \colon \mathbb{N}^{n-1} \to \mathbb{N}$ is a bijection. We must show that $g_n \colon \mathbb{N}^n \to \mathbb{N}$ is a bijection.

Let m be a positive integer. Then there exist uniquely-determined positive integers m' and j_n for which $h(m', j_n) = m$, because the function $h\colon \mathbb{N}^2 \to \mathbb{N}$ is a bijection. There then exists a unique $(n-1)$-tuple $(j_1, j_2, \ldots, j_{n-1})$ of positive integers for which $g_{n-1}(j_1, j_2, \ldots, j_{n-1}) = m'$, because $g_{n-1}\colon \mathbb{N}^{n-1} \to \mathbb{N}$ is a bijection. But then $(j_1, j_2, \ldots, j_n)$ is the unique n-tuple of positive integers for which $g_n(j_1, j_2, \ldots, j_n) = m$. We conclude therefore that $g_n\colon \mathbb{N}^n \to \mathbb{N}$ is a bijection, as required. ∎

Proposition 4.5

Let $X_1, X_2, \ldots, X_n$ be countable sets. Then the Cartesian product $X_1 \times X_2 \times \cdots \times X_n$ of these countable sets is itself a countable set.

Proof

Let $X = X_1 \times X_2 \times \cdots X_n$. There exist injective functions $f_i \colon X_i \to \mathbb{N}$ from the set X_i to the set $\mathbb{N}$ of positive integers, because each set X_i is countable. Also there exists a bijection $g_n \colon \mathbb{N}^n \to \mathbb{N}$ from the set $\mathbb{N}^n$ of ordered n-tuples of positive integers to the set $\mathbb{N}$ of positive integers (see Lemma 4.4). Let $f \colon X \to \mathbb{N}$ be defined so that

$$f(x_1, x_2, \cdots, x_n) = g_n(f_1(x_1), f_2(x_2), \ldots, f_n(x_n))$$

for all $(x_1, x_2, \ldots, x_n) \in X$. We show that $f \colon X \to \mathbb{N}$ is injective.

Let $(x_1, x_2, \ldots, x_n)$ and $(x'_1, x'_2, \ldots, x'_n)$ be elements of the set X. Suppose that

$$f(x_1, x_2, \ldots, x_n) = f(x'_1, x'_2, \ldots, x'_n).$$

Then

$$(f_1(x_1), f_2(x_2), \ldots, f_n(x_n)) = (f_1(x'_1), f_2(x'_2), \ldots, f_n(x'_n)),$$

because the function $g_n \colon \mathbb{N}^2 \to \mathbb{N}$ is injective, and therefore $f_i(x_i) = f_i(x'_i)$ for $i = 1, 2, \ldots, n$. But each of the functions $f_1, f_2, \ldots, f_n$ is injective. It follows that $x_i = x'_i$ for $i = 1, 2, \ldots, n$, and thus

$$(x_1, x_2, \ldots, x_n) = (x'_1, x'_2, \ldots, x'_n).$$

This shows that the function $f \colon X \to \mathbb{N}$ is injective. It follows that the set X is countable, as required. ∎

Proposition 4.6

Any countable union of countable sets is itself a countable set.

Proof

Let J be a subset of the set $\mathbb{N}$ of positive integers and, for each $j \in J$, let X_j be a countable set, and let $X = \bigcup_{j \in J} X_j$. Also, for each $j \in J$, let $g_j\colon X_j \to \mathbb{N}$ be an injective function from X_j to the set $\mathbb{N}$ of positive integers. (The functions g_j exist because, for all $j \in J$, the set X_j is a countable set.) For each $x \in X$ let $n(x)$ denote the smallest positive integer j in the indexing set J for which $x \in X_j$. Let $h\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be a bijection between the sets $\mathbb{N} \times \mathbb{N}$ and $\mathbb{N}$ (see Lemma 4.3), and let $f\colon X \to \mathbb{N}$ be the function defined so that

$$f(x) = h(n(x), g_{n(x)}(x))$$

for all $x \in X$.

Let x and x' be elements of X satisfying $f(x) = f(x')$. We claim that $x = x'$. Now if $f(x) = f(x')$ then $n(x) = n(x')$ and $g_{n(x)}(x) = g_{n(x')}(x')$, because the function $h\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is a bijection. Let $n = n(x)$. Then $x \in X_n$ and $x' \in X_n$, and $g_n(x) = g_n(x')$. But $g\colon X_n \to \mathbb{N}$ is an injective function. It follows that $x = x'$. We conclude therefore that the function $f\colon X \to \mathbb{N}$ is injective, and therefore the set X is countable, as required. ∎

Lemma 4.7

The set $\mathbb{Z}$ of integers is countable.

Proof

The set $\mathbb{Z}$ is the union of the set $\mathbb{N}$ of positive integers and the set W of non-positive integers, where $W = \{n \in \mathbb{Z} : n \leq 0\}$. Let $f\colon W \to \mathbb{N}$ be defined such that $f(n) = 1 - n$ for all $n \in W$. Then $f\colon W \to \mathbb{N}$ is bijective, and therefore the set W is countable. It follows that the set $\mathbb{Z}$ of integers, being the union of the countable sets $\mathbb{N}$ and W, is itself a countable set, as required. ∎

Lemma 4.8

The set $\mathbb{Q}$ of rational numbers is countable.

Proof

For each positive integer m, let R_m be the set consisting of all rational numbers that are of the form n/m for some positive integer n. The function mappng each $q \in R_m$ to mq is a bijection from R_m to the set $\mathbb{Z}$ of integers, and $\mathbb{Z}$ is a countable set. It follows that R_m is a countable set for each positive integer m. Now $\mathbb{Q} = \bigcup_{m=1}^{+\infty} R_m$. It follows that the set $\mathbb{Q}$ of rational numbers is a countable union of countable sets. Moreover any countable union of countable sets is itself countable (Proposition 4.6). We conclude that the set $\mathbb{Q}$ is countable, as required. ∎

Proposition 4.9

Let $h: X \to Y$ be a surjection. Suppose that the set X is countable. Then the set Y is countable.

Proof

The set X is countable, and therefore there exists an injective function $g\colon X \to \mathbb{N}$ from X to the set $\mathbb{N}$ of positive integers. Given any element y of the set Y there exists at least one positive integer n with the property that $n = g(x)$ for some $x \in X$ satisfying $h(x) = y$, because the function h is surjective. For each $y \in Y$, let $f(y)$ be the smallest positive integer n with the property that $n = g(x)$ for some $x \in X$ satisfying $h(x) = y$.

Let y and y' be elements of the set Y, where $y \neq y'$. Then there exist elements x and x' of the set X for which $f(y) = g(x)$, $f(y') = g(x')$, $h(x) = y$ and $h(x') = y'$. Then $x \neq x'$, because $y \neq y'$. But then $g(x) \neq g(x')$, because the function g is injective, and therefore $f(y) \neq f(y')$. We conclude from this that the function f is injective, and therefore the set Y is countable, as required. ∎

Proposition 4.10

A non-empty set X is countable if and only if there exists a surjective function $g\colon \mathbb{N} \to X$ mapping the set $\mathbb{N}$ of positive integers onto X.

Proof

Let X be a non-empty set. If there exists a surjective function $g\colon \mathbb{N} \to X$ mapping the set of positive integers onto X then it follows from Proposition 4.9 that the set X is countable.

Conversely suppose that X is a non-empty countable set. Then there exists an injection $f\colon X \to \mathbb{N}$ from X to $\mathbb{N}$. Let x_0 be some chosen element of the set X. Given a positive integer n, there exists at most one element x of the set X for which $f(x) = n$. It follows that there exists a function $g\colon \mathbb{N} \to X$, where $g(f(x)) = x$ for all $x \in X$ nd $g(n) = x_0$ for positive integers n that do not belong to the range $f(X)$ of the function f. This function $g\colon \mathbb{N} \to X$ is then a surjective function mapping the set $\mathbb{N}$ of positive integers onto the set X. The result follows. $\blacksquare$

4.3. Uncountable Sets

A set that is not countable is said to be *uncountable*. Many sets occurring in mathematics are uncountable. These include the set of real numbers.

It follows directly from Lemma 4.1 that if a set X has an uncountable subset, then X must itself be uncountable.

It also follows directly from Proposition 4.9 that if $h\colon X \to Y$ is a surjection from a set X to a set Y, and if the set Y is uncountable, then the set X is uncountable.

Definition

Let X be a set. The *power set* $\mathcal{P}(X)$ of X is the set whose elements are the subsets of the set X.

It is a straightforward exercise to prove that if a finite set X has m elements then its power set $\mathcal{P}(X)$ has 2^m elements. (This may be shown by induction on the number of elements in the finite set.) It follows that, for any finite set X, the power set $\mathcal{P}(X)$ has more elements than the set X itself, and therefore there cannot exist any surjective function from a finite set to its power set. We now show that the same is true of all sets, whether finite or infinite.

Proposition 4.11

Let X be a set, and let $\mathcal{P}(X)$ be the power set of X. Then there cannot exist any surjective function from the set X to its power set $\mathcal{P}(X)$.

Proof

Let $f\colon X \to \mathcal{P}(X)$ be a function from a set X to its power set $\mathcal{P}(X)$, and let $B = \{x \in X : x \notin f(x)\}$. Let $x \in X$. Then $x \in B$ if and only if $x \notin f(x)$. It follows that the element x of X belongs to exactly one of the subsets B and $f(x)$ of X, and therefore $B \neq f(x)$. We conclude from this that the subset B of X is an element of the power set $\mathcal{P}(X)$ of X that does not belong to the range $f(X)$ of the function f. Thus the function f is not surjective. The result follows. ∎

Corollary 4.12

The power set $\mathcal{P}(\mathbb{N})$ of the set $\mathbb{N}$ of positive integers is an uncountable set.

Proof

If the set $\mathcal{P}(\mathbb{N})$ were countable, there would exist a surjective function $f\colon \mathbb{N} \to \mathcal{P}(\mathbb{N})$ mapping the set $\mathbb{N}$ of positive integers onto its power set (see Proposition 4.10). But there cannot exist any surjective function mapping the set $\mathbb{N}$ onto its power set (Proposition 4.11). Therefore the set $\mathcal{P}(\mathbb{N})$ must be uncountable, as required. $\blacksquare$

Proposition 4.13

The set $\mathbb{R}$ of real numbers is uncountable.

Proof

Let the function $h\colon \mathcal{P}(\mathbb{N}) \to \mathbb{R}$ from the power set $\mathcal{P}(\mathbb{N})$ of the set of positive integers to the set $\mathbb{R}$ of real numbers be defined so that, for all subsets B of $\mathbb{N}$,

$$h(B) = \sum_{j=1}^{+\infty} \frac{d_j}{10^j},$$

where $d_j = 1$ whenever $j \in B$ and $d_j = 0$ whenever $j \notin B$. (Thus, for example, $\mathrm{h}(\{2,3,5,8\}) = 0.01101001$.)

The function $h\colon \mathcal{P}(\mathbb{N}) \to \mathbb{R}$ is injective. It follows that if the set $\mathbb{R}$ of real numbers were countable, then the set $\mathcal{P}(\mathbb{N})$ would also be countable (see Lemma 4.2). But the power set $\mathcal{P}(\mathbb{N})$ of the set of positive integers is uncountable (see Corollary 4.12). It follows therefore that the set $\mathbb{R}$ of real numbers is also uncountable, as required. $\blacksquare$