MA3486—Fixed Point Theorems and Economic Equilibria
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8.9. Walras's Law

In the exchange economy model under discussion, let $\mathbf{p}$ be a price vector satisfying $\mathbf{p} >> 0$, and let $\xi_h(\mathbf{p})$ be the set of bundles of commodities maximizing utility for household h, subject only to the budget constraint requiring that $\mathbf{p} \,.\, \mathbf{x} \leq \mathbf{p} \,.\, \overline{\mathbf{x}}_h$ for all bundles $\mathbf{x}$ available to household h. Then $\mathbf{p} \,.\, \mathbf{x} = \mathbf{p} \,.\, \overline{\mathbf{x}}_h$ for all $\mathbf{x} \in \xi_h(\mathbf{p})$. Summing over all households, we find that $\mathbf{p} \,.\, \mathbf{x} = \mathbf{p} \,.\, \mathbf{s}$, for all $\mathbf{x} \in \xi(\mathbf{p})$, where $\mathbf{s}$ denotes the aggregate supply, defined so that $\mathbf{s} = \sum_{h=1}^{m} \overline{\mathbf{x}}_h$, and $\xi(\mathbf{p})$ denotes the value of the aggregate demand correspondence at prices p, defined so that $\xi = \sum_{h=1}^{m} \xi_h$.

It follows that $\mathbf{p} \, . \, \mathbf{z} = 0$ for all $\mathbf{z} \in \zeta(\mathbf{p})$, where ζ denotes the *excess demand correspondence*, defined such that

$$\zeta(\mathbf{p}) = \{\mathbf{x} - \mathbf{s} : \mathbf{x} \in \xi(\mathbf{p})\}$$

for all $\mathbf{p} \in \Delta$ satisfying $\mathbf{p} >> \mathbf{0}$. This property of the excess demand correspondence is often referred to as *Walras's Law*.

8.10. Walrasian Equilibria with Strictly Quasiconcave Utility

We consider an exchange economy with n commodities and m households, retaining the notation of the previous discussion. We now consider the situation in which the utility function of each household is strictly quasiconcave.

Definition

A real-valued function $u\colon X \to \mathbb{R}$ defined on a convex subset X of $\mathbb{R}^n$ is said to be *strictly quasiconcave* on X if

$$u((1-t)\mathbf{x} + t\mathbf{x}') > \min\Big(u(\mathbf{x}), u(\mathbf{x}')\Big)$$

for all distinct points $\mathbf{x}$ and $\mathbf{x}'$ of X and for all real numbers t satisfying $0 < t < 1$.

Suppose that, in the exchange economy, the utility function u_h of household h is continuous, strictly increasing and strictly quasiconcave for $h = 1, 2, \ldots, m$. The utility function of household h cannot then be maximized at two distinct points of any non-empty compact convex set. Let $\mathbf{c}$ be an n-dimensional vector satisfying $\mathbf{c} >> 0$. Then, given any normalized price vector $\mathbf{p}$, and given an initial endowment $\overline{x}_h$ for the ith household, there is a unique bundle of commodities $\hat{\mathbf{x}}_{\mathbf{c},h}(\mathbf{p})$ satisfying the budget constraint $\mathbf{p} \,.\, \hat{\mathbf{x}}_{\mathbf{c},h}(\mathbf{p}) \leq \mathbf{p} \,.\, \overline{\mathbf{x}}_h$ and the total availability constraint $\hat{\mathbf{x}}_{\mathbf{c},h}(\mathbf{p}) \leq \mathbf{c}$ which maximizes the utility function for household h for all bundles of commodities that satisfy the budget constraint and the total availability constraint. Moreover if $\hat{\mathbf{x}}_{\mathbf{c},h}(\mathbf{p}) << \mathbf{c}$ then $\mathbf{p} \,.\, \hat{\mathbf{x}}_{\mathbf{c},h}(\mathbf{p}) = \mathbf{p} \,.\, \overline{\mathbf{x}}_h$.

The preferences of household h, given normalized prices, given its initial endowment, and given the upper bounds on the availability of each commodity specified by the components of the vector $\mathbf{c}$, therefore determine a *demand function* $\hat{\mathbf{x}}_{\mathbf{c},h} \colon \Delta \to \mathbb{R}^+_-$ on the price simplex Δ, where

$$\begin{aligned} \Delta &= \{(p_1, p_2, \ldots, p_n) \in \mathbb{R}^n : \\ &\qquad p_i \geq 0 \text{ for } i = 1, 2, \ldots, n, \text{ and } \sum_{i=1}^{n} p_i = 1\}. \end{aligned}$$

The results obtained in more generality for demand correspondences, using Berge's Maximum Theorem, ensure that this demand function $\hat{\mathbf{x}}_{\mathbf{c},h}$ is continuous on Δ.

Summing the demand functions for the households, and subtracting the initial endowments, we obtain an *excess demand function* $\hat{\mathbf{z}}_{\mathbf{c}} \colon \Delta \to \mathbb{R}^n$ on the price simplex Δ whose value at normalized prices $\mathbf{p}$ specifies the excess demand for the commodities traded, when each household seeks to purchase commodities to maximize its utility, subject to the budget constraint determined by the prices and its initial endowment, and subject to the availability constraint that no household can purchase an amount of the ith commodity exceeding in amount the ith component of the vector $\mathbf{c}$. This excess demand function on the price simplex Δ is continuous, and satisfies $\mathbf{p} \,.\, \hat{z}_{\mathbf{c}}(\mathbf{p}) \leq 0$ for all $\mathbf{p} \in \Delta$.

The existence of Walrasian equilibria at which supply at least matches demand can then be established on the basis of the following proposition, whose proof makes use of the Brouwer Fixed Point Theorem.

Proposition 8.12

Let

$$\begin{aligned}\Delta &= \{(p_1, p_2, \ldots, p_n) \in \mathbb{R}^n : \\ &\qquad p_i \geq 0 \text{ for } i = 1, 2, \ldots, n, \text{ and } \sum_{i=1}^{n} p_i = 1\},\end{aligned}$$

let $\mathbf{z}\colon \Delta \to \mathbb{R}^n$ *be a continuous function mapping* Δ *into* $\mathbb{R}^n$*, and let*

$$\mathbf{z}(\mathbf{p}) = (z_1(\mathbf{p}), z_2(\mathbf{p}), \ldots, z_n(\mathbf{p}))$$

for all $\mathbf{p} \in \Delta$*. Suppose that* $\mathbf{p} \,.\, \mathbf{z}(\mathbf{p}) \leq 0$ *for all* $\mathbf{p} \in \Delta$*. Then there exists* $\mathbf{p}^* \in \Delta$ *such that* $z_i(\mathbf{p}^*) \leq 0$ *for* $i = 1, 2, \ldots, n$*.*

Proof

Let $\mathbf{v}\colon \Delta \to \mathbb{R}^n$ be the function with ith component v_i given by

$$v_i(\mathbf{p}) = \begin{cases} p_i + z_i(\mathbf{p}) & \text{if } z_i(\mathbf{p}) > 0; \\ p_i & \text{if } z_i(\mathbf{p}) \leq 0. \end{cases}$$

Note that $\mathbf{v}(\mathbf{p}) \neq \mathbf{0}$ and the components of $\mathbf{v}(\mathbf{p})$ are non-negative for all $\mathbf{p} \in \Delta$. It follows that there is a well-defined map $\varphi\colon \Delta \to \Delta$ given by

$$\varphi(\mathbf{p}) = \frac{1}{\sum_{i=1}^{n} v_i(\mathbf{p})} \mathbf{v}(\mathbf{p}),$$

The Brouwer Fixed Point Theorem (Theorem 5.3) ensures that there exists $\mathbf{p}^* \in \Delta$ satisfying $\varphi(\mathbf{p}^*) = \mathbf{p}^*$. Then $\mathbf{v}(\mathbf{p}^*) = \lambda \mathbf{p}^*$ for some $\lambda \geq 1$. We claim that $\lambda = 1$.

Suppose that it were the case that $\lambda > 1$. Then $v_i(\mathbf{p}^*) > p_i^*$, and thus $z_i(\mathbf{p}^*) > 0$ whenever $p_i^* > 0$. But $p_i^* \geq 0$ for all i, and $p_i^* > 0$ for at least one value of i, since $\mathbf{p}^* \in \Delta$. It would follow that $\mathbf{p}^* \,.\, \mathbf{z}(\mathbf{p}^*) > 0$, contradicting the requirement that $\mathbf{p} \,.\, \mathbf{z}(\mathbf{p}) \leq 0$ for all $\mathbf{p} \in \Delta$. We conclude that $\lambda = 1$, and thus $v_i = p_i^*$ and $z_i(\mathbf{p}^*) \leq 0$ for all i, as required. ∎

8.11. Historical Note

The proof of the existence of Walrasian equilibria in exchange economies can be generalized to *Arrow-Debreu* models where economic activity is carried out by both households and firms. The problem of existence of equilibria was studied by L. Walras in the 1870s, though a rigorous proof of the existence of equilibria was not found till the 1930s, when A. Wald proved existence for a limited range of economic models. Proofs of existence using topological fixed point theorems such as the Brouwer Fixed Point Theorem or the Kakutani Fixed Point Theorem were first published in 1954 by K. J. Arrow and G. Debreu and by L. McKenzie. Subsequent research has centred on problems of uniqueness and stability, and the existence theorems have been generalized to economies with an infinite number of commodities and economic agents (households and firms). An alternative approach to the existence theorems using techniques of differential topology was pioneered by G. Debreu and by S. Smale.

More detailed accounts of the theory of 'general equilibrium' can be found in, for example, *The theory of value*, by G. Debreu, *General competitive analysis*, by K. J. Arrow and F. H. Hahn, or *Economics for mathematicians* by J. W. S. Cassels.