MA3486—Fixed Point Theorems and Economic Equilibria
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8.8. Walrasian Equilibria in Exchange Economies

Theorem 8.10

Let n be a positive integer, let

$$\Delta = \left\{ \mathbf{p} \in \mathbb{R}^n : \mathbf{p} \geq \mathbf{0} \textit{ and } \sum_{i=1}^{n} (\mathbf{p})_i = 1 \right\},$$

let K be a compact subset of $\mathbb{R}^n$, and let $\zeta \colon \Delta \rightrightarrows K$ be an upper hemicontinuous correspondence mapping points of the simplex Δ to non-empty closed convex subsets of K. Suppose that $\mathbf{p} \,.\, \mathbf{z} \leq 0$ for all $\mathbf{p} \in \Delta$ and $\mathbf{z} \in \zeta(\mathbf{p})$. Then there exist $\mathbf{p}^* \in \Delta$ and $\mathbf{z}^* \in \zeta(\mathbf{p}^*)$ for which $\mathbf{z}^* \leq \mathbf{0}$.

Proof

The set K is clearly non-empty. We may assume, without loss of generality, that the set K is both compact and convex, because if K were not convex, then it could be replaced by a compact convex set containing it.

Let $\gamma\colon \mathbb{R}^n \to \mathbb{R}$ be the function defined so that, for each $\mathbf{x} \in \mathbb{R}^n$, $\gamma(\mathbf{x})$ is the maximum of the components of $\mathbf{x}$, and let $\mu\colon \mathbb{R}^n \rightrightarrows \Delta$ be the correspondence defined such that

$$\mu(\mathbf{x}) = \{\mathbf{p} \in \Delta : \mathbf{p} \,.\, \mathbf{z} = \gamma(\mathbf{z})\}.$$

It was shown in Proposition 8.3 that the correspondence $\mu\colon \mathbb{R}^n \rightrightarrows \Delta$ is upper hemicontinuous, and $\mu(\mathbf{x})$ is a non-empty compact convex subset of Δ for all $\mathbf{x} \in \mathbb{R}^n$. Moreover $\mathbf{p} \,.\, \mathbf{x} \leq \mathbf{p}' \,.\, \mathbf{x} = \gamma(\mathbf{x})$ for all $\mathbf{p} \in \Delta$ and $\mathbf{p}' \in \mu(\mathbf{x})$. (The upper hemicontinuity of μ also follows directly on applying Berge's Maximum Theorem, which is Theorem 2.23 above.)

Let $\Phi\colon \Delta \times K \rightrightarrows \Delta \times K$ be the correspondence defined such that

$$\Phi(\mathbf{p}, \mathbf{z}) = (\mu(\mathbf{z}), \zeta(\mathbf{p}))$$

for all $\mathbf{p} \in \Delta$ and $\mathbf{z} \in K$. The correspondences μ and ζ are upper hemicontinuous and closed-valued, and every upper hemicontinuous closed-valued correspondence has a closed graph (Proposition 2.11). It follows that the correspondence Φ has closed graph. Moreover $\Phi(\mathbf{p}, \mathbf{z})$ is a non-empty closed convex subset of the compact convex set $\Delta \times K$ for all $\mathbf{p} \in \Delta$ and $\mathbf{z} \in K$. It follows from the Kakutani Fixed Point Theorem (Theorem 5.4) that there exists $(\mathbf{p}^*, \mathbf{z}^*) \in \Delta \times K$ for which $(\mathbf{p}^*, \mathbf{z}^*) \in \Phi(\mathbf{p}^*, \mathbf{z}^*)$. Then $\mathbf{p}^* \in \mu(\mathbf{z}^*)$ and $\mathbf{z}^* \in \zeta(\mathbf{p}^*)$.

Now the conditions of the theorem require that $\mathbf{p}^* \,.\, \mathbf{z} \leq 0$ for all $\mathbf{z} \in \zeta(\mathbf{p}^*)$. Combining this inequality with the definition of the correspondence μ, and noting that $\mathbf{p}^* \in \mu(\mathbf{z}^*)$ and $\mathbf{z}^* \in \zeta(\mathbf{p}^*)$, we find that

$$\mathbf{p} \,.\, \mathbf{z}^* \leq \mathbf{p}^* \,.\, \mathbf{z}^* \leq 0$$

for all $\mathbf{p} \in \Delta$. Applying this result when $\mathbf{p}$ is the vertex of Δ whose ith component is equal to 1 and whose other components are zero, we find that $(\mathbf{z}^*)_i \leq 0$ for $i = 1, 2, \ldots, n$, and thus $\mathbf{z}^* \leq \mathbf{0}$, as required. ∎

Remark

For Theorem 8.10, and its proof, see Gérard Debreu, *Theory of Value* (Cowles Foundation Monograph 17, 1959), Section 5.6. In his notes on Chapter 5 of that monograph, Debreu notes that the result was obtained and published independently by D. Gale (published 1955) and H. Nikaido (published 1956). Debreu also thanks A. Borel, P. Samuel and A. Weil for conversations that he had with them on an early formulation of the result.

Theorem 8.11

Suppose that, in a model of an exchange economy with n goods and m households, every household receives a strictly positive initial endowment of every commodity, so that the initial endowment vector $\overline{\mathbf{x}}_h$ of household h satisfies $\overline{\mathbf{x}}_h >> \mathbf{0}$ for $h = 1, 2, \ldots, m$. Suppose also that the preferences of household h are determined by a utility function u_h that is continuous, strictly increasing and quasiconcave. Then there exists a normalized price vector $\mathbf{p}^*$ satisfying $\mathbf{p}^* >> \mathbf{0}$ and, for each household h, a corresponding bundle $\mathbf{x}_h^*$ of commodities that maximizes utility for that household subject to the affordability constraint $\mathbf{p} \,.\, \mathbf{x}_h^* \leq \mathbf{p} \,.\, \overline{\mathbf{x}}_h$, so that the total supply is redistributed amongst the households, and thus

$$\sum_{h=1}^{m} \mathbf{x}_h^* = \sum_{h=1}^{m} \overline{\mathbf{x}}_h.$$

Proof

Let $\mathbf{s} = \sum_{h=1}^{m} \mathbf{x}_h$, and let $\mathbf{c} \in \mathbb{R}^n$ be chosen so that $\mathbf{c} >> \mathbf{s}$. Let

$$\Delta = \left\{ \mathbf{p} \in \mathbb{R}^n : \mathbf{p} \geq \mathbf{0} \text{ and } \sum_{i=1}^{n} (\mathbf{p})_i = 1 \right\},$$

and, for each household, let $\hat{\xi}_{\mathbf{c},h} \colon \Delta \rightrightarrows \mathbb{R}^n_+$ be the demand correspondence that sends each normalized price vector $\mathbf{p}$ in Δ to the set $\hat{\xi}_{\mathbf{c},h}(\mathbf{p})$ of bundles of commodities that maximize utility for household h subject to the affordability constraint $\mathbf{p}^* \,.\, \mathbf{x}_h \leq \mathbf{p}^* \,.\, \overline{\mathbf{x}}_h$, and the additional constraint $\mathbf{0} \leq \mathbf{x} \leq \mathbf{c}$.

Let the correspondence $\hat{\xi}_{\mathbf{c}}\colon \Delta \rightrightarrows \mathbb{R}^n_+$ be defined so that $\hat{\xi}_{\mathbf{c}} = \sum_{h=1}^{m} \hat{\xi}_{\mathbf{c},h}$. Then the correspondence $\hat{\xi}_{\mathbf{c}}$ is upper hemicontinuous and maps each normalized price vector in Δ to a non-empty compact convex subset of $\mathbb{R}^n_+$ whose elements $\mathbf{x}$ satisfy $\mathbf{p} \,.\, \mathbf{x} \leq \mathbf{p} \,.\, \mathbf{s}$ (see Corollary 8.9).

Let the correspondence $\zeta_{\mathbf{c}}\colon \Delta \to \mathbb{R}^n$ be defined so that

$$\zeta_{\mathbf{c}} = \{\mathbf{x} - \mathbf{s} : \mathbf{x} \in \hat{\xi}_{\mathbf{c}}(\mathbf{p})\}$$

for all $\mathbf{p} \in \Delta$. Then $\mathbf{p} \,.\, \mathbf{z} \leq 0$ for all $\mathbf{p} \in \Delta$ and $\mathbf{z} \in \zeta(\mathbf{p})$. Moreover $\zeta_{\mathbf{c}}$ maps Δ into the compact set

$$\{\mathbf{z} \in \mathbb{R}^n : -\mathbf{s} \leq \mathbf{z} \leq \mathbf{c} - \mathbf{s}\}.$$

It then follows from Theorem 8.10 that there exist $\mathbf{p}^* \in \Delta$ and $\mathbf{z}^* \in \zeta_{\mathbf{c}}(\mathbf{p}^*)$ for which $\mathbf{z}^* \leq \mathbf{0}$.

Now $\mathbf{z}^* + \mathbf{s} \in \hat{\xi}_{\mathbf{c}}(\mathbf{p}^*)$. It follows from the definition of $\hat{\xi}_{\mathbf{c}}(\mathbf{p}^*)$. that there exist $\mathbf{x}_h^* \in \hat{\xi}_{\mathbf{c},h}(\mathbf{p}^*)$ for $h = 1, 2, \ldots, n$ for which $\sum_{h=1}^{m} \mathbf{x}_h^* = \mathbf{z}^* + \mathbf{s}$. Then $\sum_{h=1}^{m} \mathbf{x}_h^* \leq \mathbf{s}$, because $\mathbf{z}^* \leq \mathbf{0}$. Now $\mathbf{x}_h^* \geq \mathbf{0}$ for $h = 1, 2, \ldots, m$. It follows that $\mathbf{0} \leq \mathbf{x}_h^* \leq \mathbf{s}$ and therefore $\mathbf{x}_h^* << \mathbf{c}$ for $h = 1, 2, \ldots, m$.

Now $\mathbf{x}_h^*$ maximizes the utility function u_h on the set $B_{\mathbf{c},h}(\mathbf{p}^*)$, where

$$B_{\mathbf{c},h}(\mathbf{p}^*) = \{\mathbf{x} \in \mathbb{R}^n_+ : \mathbf{0} \leq \mathbf{x} \leq \mathbf{c} \text{ and } \mathbf{p}^* \,.\, \mathbf{x} \leq \mathbf{p}^* \,.\, \overline{\mathbf{x}}_h\}.$$

Let

$$B_h(\mathbf{p}^*) = \{\mathbf{x} \in \mathbb{R}^n_+ : \mathbf{x} \geq \mathbf{0} \text{ and } \mathbf{p}^* \,.\, \mathbf{x} \leq \mathbf{p}^* \,.\, \overline{\mathbf{x}}_h\}.$$

and let

$$N = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} << \mathbf{c}\}.$$

Then the set N is open in $\mathbb{R}^n$, $\mathbf{x}_h^* \in N$ and the maximum value of the utility function u_h for household h on $B_h(\mathbf{p}^*) \cap N$ is achieved at $\mathbf{x}_h^*$. It follows directly from Proposition 8.4 that

$$\mathbf{p}^* \,.\, \mathbf{x}_h^* = \mathbf{p}^* \,.\, \overline{\mathbf{x}}_h,$$

and moreover the maximum value of the utility function u_h for household h on $B_h(\mathbf{p}^*)$ is achieved at $\mathbf{x}_h^*$.

Next we note that were it the case that $(\mathbf{p}^*)_i = 0$ for some index i between 1 and n then the amount of the ith commodity in the bundle $\mathbf{x}_h^*$ could be increased to obtain a bundle $\mathbf{x}$ for which $\mathbf{x} \neq \mathbf{x}_h^*$, $\mathbf{x} >> \mathbf{x}_h$ and $\mathbf{p}^* \,.\, \mathbf{x} = \mathbf{p}^* \,.\, \mathbf{x}_h^*$. But then $u_h(\mathbf{x}) > u_h(\mathbf{x}_h^*)$, because the utility function u_h is strictly increasing, and thus $\mathbf{x}_h^*$ would not maximize utility for for household h subject to the affordability constraint. We conclude therefore that $\mathbf{p}^* >> 0$.

Finally we note that

$$\mathbf{s} - \sum_{h=1}^{m} \mathbf{x}_h^* \geq \mathbf{0}$$

and

$$\mathbf{p}^* \,.\, \left(\mathbf{s} - \sum_{h=1}^{m} \mathbf{x}_h^* \right) = \sum_{h=1}^{m} \mathbf{p}^* \,.\, (\overline{\mathbf{x}}_h - \mathbf{x}_h^*) = 0.$$

It follows that

$$\mathbf{s} = \sum_{h=1}^{m} \mathbf{x}_h^*.$$

This completes the proof. ∎