MA3486—Fixed Point Theorems and Economic Equilibria
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Proposition 8.2

Let n be a positive integer, and let $B\colon \mathbb{R}^n_+ \times \mathbb{R}_+ \rightrightarrows \mathbb{R}^n$ be the budget correspondence that assigns to each price-wealth pair $(\mathbf{p}, w)$ in $\mathbb{R}^n_+ \times \mathbb{R}_+$ the subset $B(\mathbf{p}, w)$ of $\mathbb{R}^n_+$ defined such that

$$B(\mathbf{p}, w) = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} \geq \mathbf{0} \text{ and } \mathbf{p} \,.\, \mathbf{x} \leq w\}.$$

Then the budget correspondence $B\colon \mathbb{R}^n_+ \times \mathbb{R}_+ \rightrightarrows \mathbb{R}^n$ is both upper hemicontinuous and lower hemicontinuous on the set Γ^n, where

$$\Gamma^n = \{(\mathbf{p}, w) \in \mathbb{R}^n \times \mathbb{R} : \mathbf{p} >> \mathbf{0} \text{ and } w > 0\}.$$

Moreover $B(\mathbf{p}, w)$ of $\mathbb{R}^n_+$ is non-empty, compact and convex for all $(\mathbf{p}, w) \in \Gamma^n$.

Proof

Let $(\mathbf{p}_0, w_0)$ be a price-wealth pair for which $\mathbf{p}_0 >> \mathbf{0}$ and $w_0 > 0$. Then $(\mathbf{p})_i > 0$ for $i = 1, 2, \ldots, n$. Let a positive vector $\mathbf{c}$ be chosen so that

$$(\mathbf{c})_i > \frac{w_0}{(\mathbf{p}_0)_i}$$

for $i = 1, 2, \ldots, n$. Let

$$N = \{(\mathbf{p}, w) \in \mathbb{R}^n_+ \times \mathbb{R}_+ : w > 0 \text{ and } (\mathbf{p})_i > \frac{w}{(\mathbf{c})_i} \text{ for } i = 1, 2, \ldots, n\}.$$

Then N is an open subset of $\mathbb{R}^n_+ \times \mathbb{R}_+$, $(\mathbf{p}_0, w_0) \in N$. Moreover if $(\mathbf{p}, w) \in N$, and if $\mathbf{x} \in B(\mathbf{p}, w)$, then $\mathbf{x} \geq 0$, $\mathbf{p} \,.\, \mathbf{x} \leq w$ and $w > 0$ But then $(\mathbf{p})_i > 0$ and

$$(\mathbf{p})_i(\mathbf{x})_i \leq w < (\mathbf{p})_i(\mathbf{c})_i$$

for $i = 1, 2, \ldots, n$, and therefore $\mathbf{x} \leq \mathbf{c}$.

It follows that $B(\mathbf{p}, w) = B_{\mathbf{c}}(\mathbf{p}, w)$ for all $(\mathbf{p}, w) \in N$, where

$$B_{\mathbf{c}}(\mathbf{p}, w) = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{0} \leq \mathbf{x} \leq \mathbf{c} \text{ and } \mathbf{p} \,.\, \mathbf{x} \leq w\}.$$

Now the correspondence $B_{\mathbf{c}}$ defined in this fashion is both upper hemicontinuous and lower hemicontinuous on the set of all price-wealth pairs $(\mathbf{p}, w)$ for which $w > 0$. (Proposition 8.1). It follows that, because $w > 0$ and $B(\mathbf{p}, w) = B_{\mathbf{c}}(\mathbf{p}, w)$ for all $(\mathbf{p}, w) \in N$, the budget correspondence B is both upper hemicontinuous and lower hemicontinuous on the open subset N of the set of price-wealth pairs, and is therefore both upper and lower hemicontinuous around the price-wealth pair $(\mathbf{p}_0, w_0)$. The result follows. ∎

8.3. Maximizing Normalized Commodity Prices

Proposition 8.3

Let n be a positive integer, let

$$\Delta = \left\{ \mathbf{p} \in \mathbb{R}^n : \mathbf{p} \geq \mathbf{0} \text{ and } \sum_{i=1}^{n} (\mathbf{p})_i = 1 \right\}.$$

Let $\gamma \colon \mathbb{R}^n \to \mathbb{R}$ be the function defined so that, for each $\mathbf{x} \in \mathbb{R}^n$, $\gamma(\mathbf{x})$ is the maximum of the components of $\mathbf{x}$, and let $\mu \colon \mathbb{R}^n \rightrightarrows \Delta$ be the correspondence defined such that

$$\mu(\mathbf{x}) = \{\mathbf{p} \in \Delta : \mathbf{p} \,.\, \mathbf{x} = \gamma(\mathbf{x})\}.$$

Then the correspondence $\mu \colon \mathbb{R}^n \rightrightarrows \Delta$ is upper hemicontinuous, and $\mu(\mathbf{x})$ is a non-empty compact convex subset of Δ for all $\mathbf{x} \in \mathbb{R}^n$. Also $\mathbf{p} \,.\, \mathbf{x} \leq \mathbf{p}' \,.\, \mathbf{x} = \gamma(\mathbf{x})$ for all $\mathbf{p} \in \Delta$ and $\mathbf{p}' \in \mu(\mathbf{x})$.

Proof

Let $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{p} \in \Delta$, and let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{p} = (p_1, p_2, \ldots, p_n)$. Then $p_i \geq 0$ for $i = 1, 2, \ldots, n$, and

$$\gamma(\mathbf{x}) = \max(x_1, x_2, \ldots, x_n).$$

Let $I(\mathbf{x})$ denote those integers i between 1 and n for which $x_i = \gamma(\mathbf{x})$. Now $0 \leq p_i \leq 1$ for $i = 1, 2, \ldots, n$, and $\sum_{i=1}^{n} p_i = 1$. It follows that

$$\mathbf{p} \,.\, \mathbf{x} = \sum_{i=1}^{n} p_i x_i \leq \gamma(\mathbf{x}) \sum_{i=1}^{n} p_i = \gamma(\mathbf{x}).$$

Moreover if $x_i < \gamma(\mathbf{x})$ and $p_i > 0$ for some integer i between 1 and n then $\mathbf{p} \,.\, \mathbf{x} < \mu(\mathbf{x})$. It follows that $\mathbf{p} \,.\, \mathbf{x} \leq \gamma(\mathbf{x})$ for all $\mathbf{p} \in \Delta$, and $\mathbf{p} \,.\, \mathbf{x} = \gamma(\mathbf{x})$ if and only if $p_i = 0$ for those integers i between 1 and n for which $x_i < \gamma(\mathbf{x})$. It follows that $\mathbf{p} \,.\, \mathbf{x} = \gamma(\mathbf{x})$ if and only if $p_i = 0$ for those integers i between 1 and n for which $i \notin I(\mathbf{x})$.

Therefore

$$\begin{aligned} \mu(\mathbf{x}) &= \{(p_1, p_2, \ldots, p_n) \in \Delta : p_i = 0 \text{ whenever } (\mathbf{x})_i < \gamma(\mathbf{x})\} \\ &= \{(p_1, p_2, \ldots, p_n) \in \Delta : p_i = 0 \text{ whenever } i \notin I(\mathbf{x})\}. \end{aligned}$$

It follows that, for all $\mathbf{x} \in \mathbb{R}$, the set $\mu(\mathbf{x})$ is a closed subset of the simplex Δ, and is therefore a compact set. It is clearly non-empty and convex. Also

$$\mathbf{p} \,.\, \mathbf{x} \leq \mu(\mathbf{x}) = \mathbf{p}' \,.\, \mathbf{x}$$

for all $\mathbf{p} \in \Delta$ and $\mathbf{p}' \in \mu(\mathbf{x})$.

Let $\mathbf{x}' \in \mathbb{R}^n$, and let $\mathbf{x}' = (x_1', x_2', \ldots, x_n')$. If $i \in I(\mathbf{x}')$ then $x_i' = \gamma(\mathbf{x}')$, and if $i \notin I(\mathbf{x}')$ then $x_i' < \gamma(\mathbf{x}')$. There then exists a real number θ such that $\theta < \gamma(\mathbf{x}')$ and $x_i' < \theta$ whenever $i \notin I(\mathbf{x}')$. Let N be the subset of $\mathbb{R}^n$ consisting of those elements $(x_1, x_2, \ldots, x_n)$ of $\mathbb{R}^n$ satisfying the following two conditions:

- $x_i > \theta$ if $i \in I(\mathbf{x}')$;
- $x_i < \theta$ if $i \notin I(\mathbf{x}')$.

Then N is open in $\mathbb{R}^n$ and $\mathbf{x}' \in N$. Moreover $I(\mathbf{x}) \subset I(\mathbf{x}')$ for all $\mathbf{x} \in N$, and therefore $\mu(\mathbf{x}) \subset \mu(\mathbf{x}')$ for all $\mathbf{x} \in N$. Thus if V is open in $\mathbb{R}^n$ and if $\mu(\mathbf{x}') \subset V$ then $\mu(\mathbf{x}) \subset V$ for all $\mathbf{x} \in N$. We conclude from this that the correspondence $\mu\colon \mathbb{R}^n \to \Delta$ is upper hemicontinuous on $\mathbb{R}$. This completes the proof. $\blacksquare$

Remark

Let $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ is the standard basis of $\mathbb{R}^n$, defined so that, for each integer i between 1 and n, the ith component of $\mathbf{e}_i$ is equal to 1 and the other components are zero. Then the simplex Δ is an $(n-1)$-dimensional simplex with vertices $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$, and, for each $\mathbf{x} \in \mathbb{R}^n$, the subset $\mu(\mathbf{x})$ of Δ is the face of the simplex Δ spanned by those vertices $\mathbf{e}_i$ of Δ for which $(\mathbf{x})_i = \gamma(\mathbf{x})$, where $\gamma(\mathbf{x})$ denotes the maximum value of the components of the vector $\mathbf{x}$.

8.4. Consumer Preferences

We next discuss how each household sets out to determine its purchase requirements.

We suppose that the preferences of household h are represented by a *utility function* $u_h \colon \mathbb{R}^n_+ \to \mathbb{R}$ that is continuous, strictly increasing and quasiconcave. Such a utility function therefore satisfies the following conditions:

- the function $u \colon \mathbb{R}^n_+ \to \mathbb{R}$ is *continuous*;
- the function $u \colon \mathbb{R}^n_+ \to \mathbb{R}$ is *strictly increasing*, and thus if $\mathbf{x}, \mathbf{x}' \in \mathbb{R}^n_+$ satisfy $\mathbf{x} \leq \mathbf{x}'$ and $\mathbf{x} \neq \mathbf{x}'$ then $u(\mathbf{x}) < u(\mathbf{x}')$;
- the function $u \colon \mathbb{R}^n_+ \to \mathbb{R}$ is *quasiconcave*, and thus

$$u((1-t)\mathbf{x} + t\mathbf{x}') \geq \min\Big(u(\mathbf{x}), u(\mathbf{x}')\Big)$$

  for all $\mathbf{x}, \mathbf{x}' \in \mathbb{R}^n_+$ and $t \in [0, 1]$.

Proposition 8.4

Let $u\colon X \to \mathbb{R}$ be a function defined on a closed convex subset X of $\mathbb{R}^n$ that is continuous, strictly increasing and quasiconcave, let $\mathbf{p}$ be a non-zero non-negative price vector in $\mathbb{R}^n$, let w be a positive real number, let

$$B(\mathbf{p}, w) = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} \geq \mathbf{0} \text{ and } \mathbf{p} \,.\, \mathbf{x} \leq w\}$$

and let $\mathbf{x}^* \in B(\mathbf{p}, w)$. Suppose that there exists some open neighbourhood N of $\mathbf{x}^*$ in $\mathbb{R}^n_+$ with the property that $u(\mathbf{x}) \leq u(\mathbf{x}^*)$ for all $\mathbf{x} \in B(\mathbf{p}, w) \cap N$. Then $\mathbf{p} \,.\, \mathbf{x}^* = w$ and $u(\mathbf{x}) \leq u(\mathbf{x}^*)$ for all $\mathbf{x} \in B(\mathbf{p}, w)$.

Proof

Suppose that it were the case that $\mathbf{p} \,.\, \mathbf{x}^* < w$. Then it would be possible to find $\mathbf{x} \in N$ satisfying $\mathbf{x} >> \mathbf{x}^*$ and $\mathbf{p} \,.\, \mathbf{x} < w$. Then $\mathbf{x} \in B(\mathbf{p}, w) \cap N$. The strictly increasing property of the utility function u would then ensure that $u(\mathbf{x}) > u(\mathbf{x}^*)$. But this would contradict that assumption that the maximum of the utility function u on $B(\mathbf{p}, w) \cap N$ is attained at $\mathbf{x}^*$.

Next suppose that there were to exist in the set $B(\mathbf{p}, w)$ a commodity bundle $\mathbf{x}'$ for which $u(\mathbf{x}') > u(\mathbf{x}^*)$. It would then follow from the continuity of the utility function u that the value of utility function u would exceed $u(\mathbf{x}^*)$ throughout some open ball of positive radius centred on $\mathbf{x}'$. Now $w > 0$, and therefore $B(\mathbf{p}, w)$ has non-empty interior. Moreover every open ball of positive radius about an element of $B(\mathbf{p}, w)$ would intersect the interior of this set. It follows that there would exist a commodity bundle $\mathbf{x}''$ in the interior of $B(\mathbf{p}, w)$ lying sufficiently close to $\mathbf{x}'$ to ensure that $u(\mathbf{x}'') > u(\mathbf{x}^*)$ and $\mathbf{p} \,.\, \mathbf{x}'' < w$. The quasiconcavity of the utility function would ensure that the utility function u would take values no less than $u(\mathbf{x}^*)$ along the line segment joining the commodity bundles $\mathbf{x}^*$ and $\mathbf{x}''$. Moreover this line segment would be wholly contained within the convex set $B(\mathbf{p}, w)$.

Now $\mathbf{x}^* \in N$. Therefore there would then exist a commodity bundle $\mathbf{x}'''$ on the line segment joining $\mathbf{x}^*$ and $\mathbf{x}''$ that was distinct from $\mathbf{x}^*$ but was close enough to $\mathbf{x}^*$ to ensure that $\mathbf{x}''' \in N$. Then $u(\mathbf{x}''') \geq u(\mathbf{x}^*)$ and $\mathbf{p} \,.\, \mathbf{x}''' < w$. There would then exist a commodity bundle $\mathbf{x}$ satisfying $\mathbf{x} \geq \mathbf{x}'''$ and $\mathbf{x} \neq \mathbf{x}'''$ for which $\mathbf{x} \in N$ and $\mathbf{p} \,.\, \mathbf{x} < w$. Then $\mathbf{x} \in B(\mathbf{p}, w) \cap N$ and

$$u(\mathbf{x}) > u(\mathbf{x}''') \geq u(\mathbf{x}^*),$$

contradicting the fact that the function u achieves is maximum value on $B(\mathbf{p}, w) \cap N$ at $\mathbf{x}^*$. We conclude therefore that the maximum value of the utility function u on $B(\mathbf{p}, w)$ is attained at the point $\mathbf{x}^*$, as required. ∎

Let Γ^n be the set of price-wealth pairs $(\mathbf{p}, w)$ for which $\mathbf{p} >> \mathbf{0}$ and $w > 0$, so that

$$\Gamma^n = \{(\mathbf{p}, w) \in \mathbb{R}^n \times \mathbb{R} : \mathbf{p} >> \mathbf{0} \text{ and } w > 0\}.$$

Then the closure $\overline{\Gamma}^n$ of Γ^n in $\mathbb{R}^n \times \mathbb{R}$ satisfies

$$\overline{\Gamma}^n = \mathbb{R}^n_+ \times \mathbb{R}_+ = \{(\mathbf{p}, w) \in \mathbb{R}^n \times \mathbb{R} : \mathbf{p} \geq \mathbf{0} \text{ and } w \geq 0\}.$$

Let $B \colon \overline{\Gamma}^n \rightrightarrows \mathbb{R}^n$ denote the budget correspondence on $\overline{\Gamma}^n$, where

$$B(\mathbf{p}, w) = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} \geq \mathbf{0} \text{ and } \mathbf{p} \,.\, \mathbf{x} \leq w\}$$

for all $(\mathbf{p}, w) \in \overline{\Gamma}^n$.

Let $u\colon \overline{\Gamma}^n \to \mathbb{R}$ be a utility function for a given consumer, defined over $\overline{\Gamma}^n$, that is continuous, strictly increasing and quasiconcave. Then the utility function u and the budget correspondence B together determine a single valued function $V\colon \Gamma^n \to \mathbb{R}$ and a correspondence $\xi\colon \Gamma^n \rightrightarrows \mathbb{R}^n_+$, where

$$V(\mathbf{p}, w) = \sup\{u(\mathbf{x}) : \mathbf{x} \in B(\mathbf{p}, w)\}$$

and

$$\xi(\mathbf{p}, w) = \sup\{\mathbf{x} \in B(\mathbf{p}, w) : u(\mathbf{x}) = V(\mathbf{p}, w)\}.$$

The function $V\colon \Gamma^n \to \mathbb{R}$ is referred to as the *indirect utility function* for the given consumer, and the correspondence $\xi\colon \Gamma^n \rightrightarrows \mathbb{R}^n_+$ is referred to as the *demand correspondence* for that consumer. The value of $V(\mathbf{p}, w)$ is the maximum utility that the consumer by purchasing a bundle of commodities that is affordable for that consumer when the commodity prices are given by the price vector $\mathbf{p}$ and the wealth of the consumer is represented by the non-negative real number w. The demand correspondence $\xi\colon \Gamma^n \rightrightarrows \mathbb{R}^n_+$ associates to a price-wealth pair $(\mathbf{p}, w)$ the set consisting of those bundles of commodities that are most desirable for the consumer with wealth w, subject to being affordable at prices $\mathbf{p}$.

Proposition 8.5

In an exchange economy with n commodities, suppose that the preferences of a given consumer are represented by a utility function $u\colon \overline{\Gamma}^n \to \mathbb{R}$, defined over the line $\overline{\Gamma}^n$ of price-wealth pairs, that is continuous, strictly increasing and quasiconcave. Then the resulting indirect utility function $V\colon \Gamma^n \to \mathbb{R}$ is continuous on the set Γ^n of price-wealth pairs $(\mathbf{p}, w)$ for which $\mathbf{p} >> \mathbf{0}$ and $w > 0$, and the demand correspondence $\xi\colon \Gamma^n \rightrightarrows \mathbb{R}^n_+$ is upper hemicontinuous and maps each price-wealth pair $(\mathbf{p}, w)$ in Γ^n to a non-empty compact convex subset of $\mathbb{R}^n_+$.

Proof

Proposition 8.2 ensures that the budget correspondence $\xi\colon \Gamma^n \rightrightarrows \mathbb{R}^n_+$ is both upper hemicontinuous and lower hemicontinuous on Γ^n. Moreover $\xi(\mathbf{p}, w)$ is a non-empty compact subset of $\mathbb{R}^n_+$ for all $(\mathbf{p}, w) \in \Gamma^n$. It follows from a direct application of Berge's Maximum Theorem (Theorem 2.23) that the indirect utility function is continuous and the demand correspondence is upper hemicontinuous and maps each price-wealth pair in Γ^n to a non-empty compact subset of $\mathbb{R}^n_+$. The convexity of $B(\mathbf{p}, w)$ and the quasiconcavity of the utility function u then ensure that $\xi(\mathbf{p}, w)$ is convex for all price-wealth pairs $(\mathbf{p}, w)$ in Γ^n. ∎

Let $\mathbf{c}$ be an element of $\mathbb{R}^n$ satisfying $\mathbf{c} >> \mathbf{0}$. In what follows we restrict consumer choice to those bundles of commodities that, for a particular price-wealth pair $(\mathbf{p}, w)$, are both affordable and subject to the availability constraint $\mathbf{0} \leq \mathbf{x} \leq \mathbf{c}$. Thus let $B_{\mathbf{c}} \colon \overline{\Gamma}^n \rightrightarrows \mathbb{R}^n$ denote the budget correspondence on $\overline{\Gamma}^n$ when availability is constrained in this fashion, so that

$$B_{\mathbf{c}}(\mathbf{p}, w) = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{0} \leq \mathbf{x} \leq \mathbf{c} \text{ and } \mathbf{p} \,.\, \mathbf{x} \leq w\}$$

for all $(\mathbf{p}, w) \in \overline{\Gamma}^n$. It follows from Proposition 8.1 that the correspondence $B_{\mathbf{c}} \colon \overline{\Gamma}^n \rightrightarrows \mathbb{R}^n$ is both upper hemicontinuous and lower hemicontinuous throughout the set $\hat{\Gamma}^n$ defined so that

$$\hat{\Gamma}^n = \{(\mathbf{p}, w) \in \mathbb{R}^n \times \mathbb{R} : \mathbf{p} \geq \mathbf{0} \text{ and } w > 0\}.$$

We still require the utility function $u\colon \overline{\Gamma}^n \to \mathbb{R}$ for the given consumer to be continuous, strictly increasing and quasiconcave. Then the utility function u and the modified budget correspondence $B_{\mathbf{c}}$ together determine a single valued function $\hat{V}_{\mathbf{c}}\colon \Gamma^n \to \mathbb{R}$ and a correspondence $\hat{\xi}_{\mathbf{c}}\colon \Gamma^n \rightrightarrows \mathbb{R}^n_+$, where

$$\hat{V}_{\mathbf{c}}(\mathbf{p}, w) = \sup\{u(\mathbf{x}) : \mathbf{x} \in B_{\mathbf{c}}(\mathbf{p}, w)\}$$

and

$$\hat{\xi}_{\mathbf{c}}(\mathbf{p}, w) = \sup\{\mathbf{x} \in B_{\mathbf{c}}(\mathbf{p}, w) : u(\mathbf{x}) = \hat{V}_{\mathbf{c}}\mathbf{p}, w)\}.$$

Proposition 8.6

In an exchange economy with n commodities, suppose that the preferences of a given consumer are represented by a utility function $u\colon \overline{\Gamma}^n \to \mathbb{R}$, defined over the line $\overline{\Gamma}^n$ of price-wealth pairs, that is continuous, strictly increasing and quasiconcave. Let $\mathbf{c} \in \mathbb{R}^n$ satisfy $\mathbf{c} >> \mathbf{0}$, and let the consumer be required to select from bundles $\mathbf{x}$ of commodities, represented by non-negative n-dimensional vectors, that, for prices and wealth given by the price-wealth pair $(\mathbf{p}, w)$, satisfy both the budget constraint $\mathbf{p} \,.\, \mathbf{x} \leq w$ and the availability constraint $\mathbf{0} \leq \mathbf{x} \leq \mathbf{c}$. Then the resulting indirect utility function $\hat{V}_{\mathbf{c}}\colon \hat{\Gamma}^n \to \mathbb{R}$ is continuous on the set $\hat{\Gamma}^n$ of price-wealth pairs $(\mathbf{p}, w)$ for which $w > 0$, and the demand correspondence $\hat{\xi}_{\mathbf{c}}\colon \hat{\Gamma}^n \rightrightarrows \mathbb{R}^n_+$ is upper hemicontinuous and maps each price-wealth pair $(\mathbf{p}, w)$ in $\hat{\Gamma}^n$ to a non-empty compact convex subset of $\mathbb{R}^n_+$.

Proof

Proposition 8.1 ensures that the budget correspondence $\hat{\xi}_{\mathbf{c}} \colon \hat{\Gamma}^n \rightrightarrows \mathbb{R}^n_+$ is both upper hemicontinuous and lower hemicontinuous on $\hat{\Gamma}^n$. Moreover $\hat{\xi}_{\mathbf{c}}(\mathbf{p}, w)$ is a non-empty compact subset of $\mathbb{R}^n_+$ for all $(\mathbf{p}, w) \in \hat{\Gamma}^n$. It follows from a direct application of Berge's Maximum Theorem (Theorem 2.23) that the indirect utility function is continuous and the demand correspondence is upper hemicontinuous and maps each price-wealth pair in $\hat{\Gamma}^n$ to a non-empty compact subset of $\mathbb{R}^n_+$. The convexity of $B_{\mathbf{c}}(\mathbf{p}, w)$ and the quasiconcavity of the utility function u then ensure that $\hat{\xi}_{\mathbf{c}}(\mathbf{p}, w)$ is convex for all price-wealth pairs $(\mathbf{p}, w)$ in $\hat{\Gamma}^n$. $\blacksquare$

8.6. Addition of Compact-Valued Correspondences

We discuss now the addition of vector-valued correspondences.

Suppose that we have m correspondences $\xi_1, \xi_2, \ldots, \xi_m$ defined over some subset Ω of a Euclidean space, and mapping points of Ω to subsets of a Euclidean space $\mathbb{R}^n$. Let $\sum_{h=1}^{n} \xi_h$ denote the correspondence ξ defined such that

$$\xi(\mathbf{p}) = \left\{ \sum_{h=1}^{m} \mathbf{x}_h : \mathbf{x}_h \in \xi_h(\mathbf{p}) \right\}.$$

Proposition 8.7

Let $\xi_1, \xi_2, \ldots, \xi_m$ be correspondences defined over some subset Ω of a Euclidean space, and mapping points of that space to non-empty compact subsets of the n-dimensional Euclidean space $\mathbb{R}^n$. Suppose that these correspondences are upper hemicontinuous. Then the sum $\sum_{h=1}^{m} \xi_h$ of those correspondences is an upper hemicontinuous correspondence mapping points of Ω to non-empty compact subsets of $\mathbb{R}^n$.

Proof

Let $\xi \colon \Omega \rightrightarrows \mathbb{R}^n$ be the correspondence that is the sum $\sum_{h=1}^{m} \xi_h$ of the correspondences $\xi_1, \xi_2, \ldots, \xi_m$. Now, for each $\mathbf{p} \in \Omega$, the set $\xi(\mathbf{p})$ is the image of the Cartesian product

$$\xi_1(\mathbf{p}) \times \xi_2(\mathbf{p}) \times \cdots \times \xi_m(\mathbf{p})$$

under the continuous function that maps each m-tuple of vectors in $\mathbb{R}^n$ to the sum of its components. Moreover $\xi_h(\mathbf{p})$ is, by assumption, a non-empty compact subset of $\mathbb{R}^n$, and any Cartesian product of non-empty compact sets is non-empty and compact, and the image of a non-empty compact set under a continuous map is non-empty and compact. We conclude therefore that $\xi(\mathbf{p})$ is a non-empty compact subset of $\mathbb{R}^n$ for all $\mathbf{p} \in \Omega$.

We can therefore apply the "ϵ–δ" criterion for upper hemicontinuity of compact-valued correspondences established by Proposition 2.16. Given any subset K of $\mathbb{R}^n$, and given any positive real number r, we denote by $B(K, r)$ the subset of $\mathbb{R}^n$ that lie within a distance less than r of a point of K.

Let $\mathbf{p} \in \Omega$, and let some strictly positive real number ε be given. It follows from Proposition 2.16 that, for each integer h between 1 and m, there exists some open neighbourhood N_h of $\mathbf{p}$ in Ω such that $\xi_h(\mathbf{p}') \subset B(\xi_h(\mathbf{p}), \varepsilon/m)$ for all $\mathbf{p}' \in N_h$. Let N be the open neighbourhood of $\mathbf{p}$ in Ω that is the intersection of $N_1, N_2, \ldots, N_h$. Then a straightforward application of the triangle inequality ensures that $\xi(\mathbf{p}') \subset B(\xi(\mathbf{p}), \varepsilon)$ for all $\mathbf{p}' \in N$. It then follows from Proposition 2.16 that the correspondence $\xi \colon \Omega \rightrightarrows \mathbb{R}^n$ is upper hemicontinuous at $\mathbf{p}$. Its values are non-empty compact subsets of $\mathbb{R}^n$. The result follows. ∎

8.7. Aggregate Supply and Demand in an Exchange Economy

We now consider the properties of aggregate supply and demand in a pure exchange economy, or market, in which n commodities are traded between m households. Each household is provided with an initial endowment of commodities. The initial endowment of household h is then represented by an n-dimensional vector $\overline{\mathbf{x}}_h$ whose ith component specifies the initial endowment (relative to some appropriate unit) of the ith commodity traded in the market. The *aggregate supply* is then represented by a vector $\mathbf{s}$ that is the sum of the initial endowment vectors of all households. Thus

$$\mathbf{s} = \sum_{h=1}^{m} \overline{\mathbf{x}}_h.$$

We restrict our attention to the situation in which $\overline{\mathbf{x}}_h >> \mathbf{0}$ for $h = 1, 2, \ldots, m$. This restriction requires that each household be given an initial endowment of every commodity traded in the market. This ensures that, provided all commodity prices are non-negative, and at least one commodity price is strictly positive, then initial endowment of each household has strictly positive value, and thus each household has wealth to enable it to trade in the market. Within the mathematical model, this ensures that the demand correspondences of each household are lower hemicontinuous (see Proposition 8.6). The requirement that $\overline{\mathbf{x}}_h >> \mathbf{0}$ for all households h also ensures that $\mathbf{s} >> \mathbf{0}$.

The prices of the commodities are encoded in a price vector $\mathbf{p}$ whose components are non-negative real numbers. The ith component of this price vector $\mathbf{p}$ specifies the price of a unit of the ith commodity. We suppose that the price of at least one commodity is non-zero.

Each household seeks to trade its initial endowment for the bundle of commodities that provides it with maximum utility within the budget constraint that requires the value of purchased commodities to be less than or equal to the value of the initial endowment traded in. A consequence of this is that the demand of the *i*th consumers at prices $\lambda \mathbf{p}$ is identical to the demand at prices $\mathbf{p}$ for all positive real numbers λ. Indeed the bundles of commodities available to household h at prices $\mathbf{p}$ are those represented by vectors $\mathbf{x}_h$ satisfying the budget constraint

$$\mathbf{p} \,.\, \mathbf{x}_h \leq \mathbf{p} \,.\, \overline{\mathbf{x}_h}.$$

It follows that the price vector $\mathbf{p}$ may be replaced by the scalar multiple $\lambda \mathbf{p}$ for any positive real number λ without altering the set of bundles of commodities that the households individually can afford.

It is appropriate therefore to normalize prices in some fashion so that all non-zero non-negative price vectors can be expressed uniquely as a scalar multiple of a normalized price vector. We adopt the normalization scheme in which the sum of the prices of the commodities is required to be equal to one.

Definition

A price vector $\mathbf{p}$ (with non-negative components) is said to be *normalized* if $\sum_{i=1}^{n} (\mathbf{p})_i = 1$.

Normalized price vectors are therefore represented by the points of the *price simplex* Δ, where

$$\Delta = \left\{ \mathbf{p} \in \mathbb{R}^n : \mathbf{p} \geq \mathbf{0} \text{ and } \sum_{i=1}^{n} (\mathbf{p})_i = 1 \right\}.$$

We suppose that the demand for each household is determined by the appropriate budget constraint and by a utility function that is continuous, strictly increasing and quasiconcave. This being the case, if the price of the ith commodity is zero, with the result that the commodity is free, then every household can afford to acquire unlimited quantities of it, and because the utility functions are required to be strictly increasing, demand for that commodity cannot be satisfied: the households have an *insatiable* appetite for free commodities.

This might suggest constraining price variation to price vectors whose components are strictly positive. However the fixed point theorems that are used to prove the existence of equilibria in which supply balances demand apply to functions or correspondences defined on compact sets. Therefore the correspondences that specify the demands of the consumers as prices vary should assign a non-empty compact set not only to the normalized price vectors in the interior of the price simplex Δ but also to the price vectors on the boundary of the price simplex.

Accordingly we impose an additional constraint on the purchases of the households. In addition to the budget constraint, we place limits on the amount of each commodity in the bundles available to the households. These limits may be specified by a fixed positive vector $\mathbf{c}$. Accordingly we require that $\mathbf{c} \gg \mathbf{0}$ and that, for each integer h between 1 and m, household h selects a bundle at prices $\mathbf{p}$ to maximize utility amongst bundles $\mathbf{x}$ that satisfy both the budget constraint

$$\mathbf{p} \,.\, \mathbf{x} \leq \mathbf{p} \,.\, \overline{\mathbf{x}_h}$$

and the additional constraint $\mathbf{0} \leq \mathbf{x} \leq \mathbf{c}$.

We denote by $B_{\mathbf{c},h}(\mathbf{p})$ the set of bundles of commodities from which household $\mathbf{x}$ makes its selection. Accordingly, with this additional constraint, for each price vector $\mathbf{p}$ belonging to the price simplex Δ, household h selects the bundle of commodities that maximizes its utility function u_h over the non-empty compact set $B_{\mathbf{c},h}(\mathbf{p})$, where

$$B_{\mathbf{c},h}(\mathbf{p}) = \{\mathbf{x} \in \mathbb{R}^n_+ : \mathbf{0} \leq \mathbf{x} \leq \mathbf{c} \text{ and } \mathbf{p} \,.\, \mathbf{x} \leq \mathbf{p} \,.\, \overline{\mathbf{x}}_h\}.$$

We denote the set of bundles of commodities that maximizes utility for household h under these constraints by $\hat{\xi}_{\mathbf{c},h}(\mathbf{p})$. We obtain in this fashion a correspondence $\hat{\xi}_{\mathbf{c},h} \colon \Delta \rightrightarrows \mathbb{R}^n_+$ that determines the set of bundles maximizing utility for household h at prices $\mathbf{p}$, subject to the budget constraint and the additional constraint that available bundles of commodities by bounded above by the positive vector $\mathbf{c}$.

Proposition 8.8

Suppose that, in a model of an exchange economy with n goods and m households, every household receives a strictly positive initial endowment of every commodity, so that the initial endowment vector $\overline{\mathbf{x}}_h$ of household h satisfies $\overline{\mathbf{x}}_h >> \mathbf{0}$ for $h = 1, 2, \ldots, m$. Suppose also that the preferences of household h are determined by a utility function u_h that is continuous, strictly increasing and quasiconcave. Then, for each household, and for each $\mathbf{c} \in \mathbb{R}^n$ satisfying $\mathbf{c} >> \mathbf{0}$ the demand correspondences $\hat{\xi}_{\mathbf{c},h} \colon \Delta \rightrightarrows \mathbb{R}^n_+$ is upper hemicontinuous on the set Δ of normalized price vectors, and maps each normalized price vector $\mathbf{p}$ to a non-empty compact convex subset $\hat{\xi}_{\mathbf{c},h}(\mathbf{p})$ of $\mathbb{R}^n_+$ that consists of those bundles $\mathbf{x}$ of commodities that maximize utility for household h at prices $\mathbf{p}$ subject to both the affordability constraint $\mathbf{p} \,.\, \mathbf{x} \leq \mathbf{p} \,.\, \overline{\mathbf{x}}$ and the constraint $\mathbf{0} \leq \mathbf{x} \leq \mathbf{c}$.

Proof
Let

$$\hat{\Gamma}^n = \{(\mathbf{p}, w) \in \mathbb{R}^n \times \mathbb{R} : \mathbf{p} \geq \mathbf{0} \text{ and } w > 0\},$$

and, for all $(\mathbf{p}, w) \in \hat{\Gamma}^n$, let us denote by $\hat{\xi}'_{\mathbf{c},h}(\mathbf{p}, w)$ the demand of household h at prices $\mathbf{p}$, when the household has wealth h, where $\hat{\xi}'_{\mathbf{c},h}(\mathbf{p}, w)$ is the set of bundles $\mathbf{x}$ of commodities maximizing utility for household h at prices $\mathbf{p}$ subject to the constraints $\mathbf{p} \,.\, \mathbf{x} \leq w$ and $\mathbf{0} \leq \mathbf{x} \leq \mathbf{c}$. It follows from Proposition 8.6 that this correspondence $\hat{\xi}'_{\mathbf{c},h}$ is upper hemicontinuous on $\hat{\Gamma}^n$, and moreover it maps each price-wealth pair in $\hat{\Gamma}^n$ to a non-empty compact convex subset of $\mathbb{R}^n$. Let $\psi_h \colon \Delta \to \hat{\Gamma}^n$ be the continuous mapping that sends $\mathbf{p} \in \Delta$ to $(\mathbf{p}, \mathbf{p} \,.\, \overline{\mathbf{x}}_h)$. Then $\hat{\xi}_{\mathbf{c},h} = \hat{\xi}'_{\mathbf{c},h} \circ \psi_h$: in other words,

$$\hat{\xi}_{\mathbf{c},h}(\mathbf{p}) = \hat{\xi}'_{\mathbf{c},h}(\mathbf{p}, \mathbf{p} \,.\, \overline{\mathbf{x}}_h) = \hat{\xi}'_{\mathbf{c},h}(\psi_h(\mathbf{p})).$$

8. Walrasian Equilibria (continued)

It follows that the correspondence $\hat{\xi}_{\mathbf{c},h}\colon \Delta \to \mathbb{R}^n_+$ is the composition of a continuous mapping followed by an upper hemicontinuous correspondence. Any correspondence of this type must itself be an upper hemicontinuous correspondence. Moreover the images of normalized price vectors in Δ are subsets of $\mathbb{R}^n_+$ that have the required properties. $\blacksquare$

Now, because the demand correspondences $\hat{\xi}_{\mathbf{c},h} \colon \Delta \rightrightarrows \mathbb{R}^n_+$ for the individual households assign to each normalized price vector in the price simplex a non-empty compact subset of $\mathbb{R}^n_+$, these demand correspondences may be added together to obtain an correspondence $\hat{\xi}_{\mathbf{c}} \colon \Delta \rightrightarrows \mathbb{R}^n_+$ that represents aggregate demand from the entire economy for each normalized price vector belonging to the price simplex Δ.

An immediate application of Proposition 8.7 yields the following result.

Corollary 8.9

Suppose that, in a model of an exchange economy with n goods and m households, every household receives a strictly positive initial endowment of every commodity, so that the initial endowment vector $\overline{\mathbf{x}}_h$ of household h satisfies $\overline{\mathbf{x}}_h >> \mathbf{0}$ for $h = 1, 2, \ldots, m$. Suppose also that the preferences of household h are determined by a utility function u_h that is continuous, strictly increasing and quasiconcave. Let Δ denote the simplex whose elements are the normalized price vectors, and, for each $\mathbf{c} \in \mathbb{R}^n$ satisfying $\mathbf{c} >> \mathbf{0}$, let the demand correspondence $\hat{\xi}_{\mathbf{c},h} \colon \Delta \rightrightarrows \mathbb{R}^n_+$ be defined as specified in the statement of Proposition 8.8, let $\mathbf{s} = \sum_{h=1}^{m} \overline{\mathbf{x}}_h$, and let $\hat{\xi}_{\mathbf{c}} = \sum_{h=1}^{m} \hat{\xi}_{\mathbf{c},h}$. Then the the aggregate demand correspondence $\hat{\xi}_{\mathbf{c}} \colon \Delta \rightrightarrows \mathbb{R}^n_+$ is upper hemicontinuous on Δ, and maps each element of Δ to a non-empty compact convex subset of $\mathbb{R}^n_+$. Moreover $\mathbf{p} \,.\, \mathbf{x} \leq \mathbf{p} \,.\, \mathbf{s}$ for all $\mathbf{p} \in \Delta$ and $\mathbf{x} \in \hat{\xi}_{\mathbf{c}}(\mathbf{p})$.