MA3486—Fixed Point Theorems and Economic Equilibria
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8. Walrasian Equilibria

8.1. Exchange Economies

We consider an exchange economy consisting of a finite number of commodities and a finite number of households, each provided with an initial endowment of each of the commodities. The commodities are required to be *infinitely divisible*: this means that a household can hold an amount x of that commodity for any non-negative real number x. (Thus salt, for example, could be regarded as an 'infinitely divisible' quantity whereas cars cannot: it makes little sense to talk about a particular household owning 2.637 of a car, for example, though such a household may well own 2.637 kilograms of salt.) Now the households may well wish to exchange commodities with one another so as improve on their initial endowment. They might for example seek to barter commodities with one another: however this method of redistribution would not work very efficiently in a large economy.

Alternatively they might attempt to set up a price mechanism to simplify the task of redistributing the commodities. Thus suppose that each commodity is assigned a given price. Then each household could sell its initial endowment to the market, receiving in return the value of its initial endowment at the given prices. The household could then purchase from the market a quantity of each commodity so as to maximize its own preference, subject to the constraint that the total value of the commodities purchased by any household cannot exceed the value of its initial endowment at the given prices.

The problem of redistribution then becomes one of fixing prices so that there is exactly enough of each commodity to go around: if the price of any commodity is too low then the demand for that commodity is likely to outstrip supply, whereas if the price is too high then supply will exceed demand. A *Walras equilibrium* is achieved if prices can be found so that the supply of each commodity matches its demand. We shall use *Berge's Maximum Theorem* and the *Kakutani fixed point theorem* to prove the existence of a Walras equilibrium in this idealized economy.

Let our exchange economy consist of n commodities and m households. We suppose that household h is provided with an initial endowment $\overline{x}_{hi}$ of commodity i, where $\overline{x}_{hi} \geq 0$. Thus the initial endowment of household h can be represented by a vector $\overline{\mathbf{x}}_h$ in $\mathbb{R}^n$ whose ith component is $\overline{x}_{hi}$. The prices of the commodities are given by a price vector $\mathbf{p}$ whose ith component p_i specifies the price of a unit of the ith commodity: a price vector $\mathbf{p}$ is required to satisfy $p_i \geq 0$ for all i. Then the value of the initial endowment of household h at the given prices is $\mathbf{p} \,.\, \overline{\mathbf{x}}_h$. This quantity represents the *wealth* of household h at prices $\mathbf{p}$.

Definition

For each positive integer n, the *positive orthant* $\mathbb{R}^n_+$ is the subset of $\mathbb{R}^n$ defined so that

$$\mathbb{R}^n_+ = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} \geq \mathbf{0}\}.$$

In particular $\mathbb{R}_+ = \{t \in \mathbb{R} : t \geq 0\}$.

Definition

A real-valued function $u\colon X \to \mathbb{R}$ defined over a subset X of $\mathbb{R}^n$ is said to be *strictly increasing* on X if $u(\mathbf{x}) < u(\mathbf{x}')$ for all $\mathbf{x}, \mathbf{x}'$ in X satisfying $\mathbf{x} \leq \mathbf{x}'$ and $\mathbf{x} \neq \mathbf{x}'$.

8.2. The Budget Correspondence

We now discuss basic properties of the *budget correspondence*.

The budget correspondence is defined on the set of pairs. A *price-wealth pair* is an ordered pair $(\mathbf{p}, w)$, where $\mathbf{p} \in \mathbb{R}^n$, w is a non-negative real number and $\mathbf{p} \geq 0$. The budget correspondence assigns to each price-wealth pair the bundles of commodities that an economic agent with the specified wealth can afford to purchase at the specified prices.

More formally, the definition of the budget correspondence may be given as follows.

Definition

In a model of an exchange economy with n commodities, The *budget correspondence* $B \colon \mathbb{R}^n_+ \times \mathbb{R}_+ \rightrightarrows \mathbb{R}^n$ assigns to each price-wealth pair $(\mathbf{p}, w)$ in $\mathbb{R}^n_+ \times \mathbb{R}_+$ the subset $B(\mathbf{p}, w)$ of $\mathbb{R}^n_+$ defined such that

$$B(\mathbf{p}, w) = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} \geq \mathbf{0} \text{ and } \mathbf{p} \,.\, \mathbf{x} \leq w\}.$$

Example

Consider the case of two commodities. The budget correspondence $B\colon \mathbb{R}^2_+ \times \mathbb{R}_+ \rightrightarrows \mathbb{R}^2$ is defined so that

$$B(\mathbf{p}, w) = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0, \;\; x_2 \geq 0 \text{ and } p_1 x_1 + p_2 x_2 \leq w\}$$

for all $\mathbf{p} \in \mathbb{R}^2_+$ and $w \in \mathbb{R}_+$, where $\mathbf{p} = (p_1, p_2)$.

Let $\mathbf{p}_0$ be the vector in $\mathbb{R}^2_+$ with $\mathbf{p}_0 = (1, 0)$, and let V be the open set in $\mathbb{R}^3$ defined so that

$$V = \left\{(x_1, x_2) \in \mathbb{R}^2 : x_1 < 1 + \frac{1}{1 + x_2^2}\right\}.$$

Now

$$B(\mathbf{p}_0, w) = \{(x_1, x_2) \in \mathbb{R}^2 : 0 \leq x_1 \leq w \text{ and } x_2 \geq 0\}$$

for all $w > 0$. It follows that $B(\mathbf{p}_0, 1) \subset V$, but $B(\mathbf{p}_0, w) \not\subset V$ for all $w > 1$. Indeed if $w > 1$ then t can be chosen large enough to ensure that

$$w > 1 + \frac{1}{1 + t^2}.$$

But then $(w, t) \in B(\mathbf{p}_0, w)$, but $(w, t) \notin V$. This example demonstrates that the budget correspondence $B\colon \mathbb{R}^2_+ \times \mathbb{R}_+ \rightrightarrows \mathbb{R}^2$ is not upper hemicontinuous at $(\mathbf{p}_0, 1)$, where $\mathbf{p}_0 = (1, 0)$.

Note also that $B(\mathbf{p}, w) = B(w^{-1}\mathbf{p}, 1)$ for all $(\mathbf{p}, w) \in \mathbb{R}^2 \times \mathbb{R}_+$ satisfying $w > 0$. It follows that the budget correspondence $\mathbf{p} \mapsto B(\mathbf{p}, 1)$ is not upper hemicontinuous on $\mathbb{R}^2_+$ at $\mathbf{p}_0$.

Now let $\mathbf{p}_0 = (1, 0)$ as before, and let

$$V = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 > 1\}.$$

Now

$$B(\mathbf{p}_0, 0) = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 = 0 \text{ and } x_2 \geq 0\}.$$

It follows that $B(\mathbf{p}_0, 0) \cap V \neq \emptyset$. But if $\mathbf{p} >> \mathbf{0}$ then $B(\mathbf{p}, 0) = \{(0, 0)\}$. Thus $B(\mathbf{p}, 0) \cap V = \emptyset$ whenever $\mathbf{p} \geq \mathbf{0}$. It follows that the budget correspondence B is not lower hemicontinuous at $(\mathbf{p}_0, 0)$.

Proposition 8.1

Let n be a positive integer, let $\mathbf{c}$ be an element of $\mathbb{R}^n$ satisfying $\mathbf{c} >> \mathbf{0}$, and let $B_{\mathbf{c}}\colon \mathbb{R}^n_+ \times \mathbb{R}_+ \rightrightarrows \mathbb{R}^n$ be the correspondence that assigns to each price-wealth pair $(\mathbf{p}, w)$ in $\mathbb{R}^n_+ \times \mathbb{R}_+$ the subset $B_{\mathbf{c}}(\mathbf{p}, w)$ of $\mathbb{R}^n_+$ defined such that

$$B_{\mathbf{c}}(\mathbf{p}, w) = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{0} \leq \mathbf{x} \leq \mathbf{c} \text{ and } \mathbf{p} \,.\, \mathbf{x} \leq w\}.$$

Then the correspondence $B_{\mathbf{c}}\colon \mathbb{R}^n_+ \times \mathbb{R}_+ \rightrightarrows \mathbb{R}^n$ is upper hemicontinuous on $\mathbb{R}^n_+ \times \mathbb{R}$ and lower hemicontinuous on

$$\{(\mathbf{p}, w) \in \mathbb{R}^n_+ \times \mathbb{R} : w > 0\}.$$

Moreover $B_{\mathbf{c}}(\mathbf{p}, w)$ of $\mathbb{R}^n_+$ is non-empty, compact and convex for all $(\mathbf{p}, w) \in \mathbb{R}^n_+ \times \mathbb{R}$.

Proof

The set $B_{\mathbf{c}}(\mathbf{p}, w)$ is a non-empty closed bounded convex subset of $\mathbb{R}^n_+$ for $i = 1, 2, \ldots, n$. Any closed bounded subset of $\mathbb{R}^n$ is compact. It follows that The set $B_{\mathbf{c}}(\mathbf{p}, w)$ is non-empty, compact convex for all $(\mathbf{p}, w) \in \mathbb{R}^n_+ \times \mathbb{R}_+$.

Next we show that the correspondence $B_{\mathbf{c}}$ is upper hemicontinuous on $\mathbb{R}^n_+ \times \mathbb{R}_+$. Let $(\mathbf{p}_0, w_0) \subset \mathbb{R}^n_+ \times \mathbb{R}_+$, and let V be an open set in $\mathbb{R}^n$ for which $B_{\mathbf{c}}(\mathbf{p}_0, w_0) \subset V$. We will show that there exists an open set N in $\mathbb{R}^n_+ \times \mathbb{R}_+$ such that $(\mathbf{p}_0, w_0) \in N$ and $B_{\mathbf{c}}(\mathbf{p}, w) \subset V$ for all $(\mathbf{p}, w) \in N$.

Now $B_{\mathbf{c}}(\mathbf{p}, w) \subset C$ for all $(\mathbf{p}, w) \in \mathbb{R}^n_+ \times \mathbb{R}_+$, where

$$C = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{0} \leq \mathbf{x} \leq \mathbf{c}\}.$$

It follows that if $C \subset V$ then $B_{\mathbf{c}}(\mathbf{p}, w) \subset V$ for all $(\mathbf{p}, w) \in \mathbb{R}^n_+ \times \mathbb{R}_+$. We may therefore take $N = \mathbb{R}^n_+ \times \mathbb{R}_+$ in the case where $C \subset V$.

In the case where C is not contained in V let $F = C \setminus V$. Then F is a non-empty closed subset of C. If $\mathbf{x} \in C$ and $\mathbf{p}_0 \,.\, \mathbf{x} \leq w_0$ then $\mathbf{x} \in B_{\mathbf{c}}(\mathbf{p}_0, w_0)$, and therefore $\mathbf{x} \in V$, because $B_{\mathbf{c}}(\mathbf{p}_0, w_0) \subset V$, and thus $\mathbf{x} \notin F$. It follows that $\mathbf{p}_0 \,.\, \mathbf{x} > w_0$ for all $\mathbf{x} \in F$. It then follows from the Extreme Value Theorem that the continuous function sending each point $\mathbf{x}$ of F to $\mathbf{p}_0 \,.\, \mathbf{x}$ attains a minimum value at some point of the set F, and therefore there exists a point $\mathbf{x}_1$ of F and a real number w_1 such that $\mathbf{p}_0 \,.\, \mathbf{x}_1 = w_1$ and $\mathbf{p}_0 \,.\, \mathbf{x} \geq w_1$ for all $\mathbf{x} \in F$. Then $w_1 > w_0$. It follows that $\mathbf{p}_0 \,.\, \mathbf{x} \geq w_1$ for all $\mathbf{x} \in F$, and therefore $B_{\mathbf{c}}(\mathbf{p}_0, w_1) \cap F = \emptyset$. But $B_{\mathbf{c}}(\mathbf{p}_0, w_1) \subset C$ and $F = C \setminus V$. It follows that $B_{\mathbf{c}}(\mathbf{p}_0, w_1) \subset V$.

Now let N be the subset of $\mathbb{R}^n_+ \times \mathbb{R}^+$ consisting of those price-wealth pairs $(\mathbf{p}, w)$ with the property that

$$(\mathbf{p})_i > \frac{w}{w_1}(\mathbf{p}_0)_i$$

for those integers i between 1 and n for which $(\mathbf{p}_0)_i > 0$. Then N is open in $\mathbb{R}^n_+ \times \mathbb{R}_+$. Moreover the definition of N and the inequality $w_0 < w_1$ together ensure that $(\mathbf{p}_0, w_0) \in N$.

Care needs to be exercised in cases where $w = 0$. Suppose that $\mathbf{p} \geq 0$ and $(\mathbf{p}, 0) \in N$. Then $(\mathbf{p})_i > 0$ for all integers i between 1 and n for which $(\mathbf{p}_0)_i > 0$. It follows that if $\mathbf{x} \in \mathbb{R}^n$ satisfies $\mathbf{x} \geq \mathbf{0}$ and $\mathbf{p} \,.\, \mathbf{x} = 0$ then $(\mathbf{p})_i = 0$ for those integers i between 1 and n for which $(\mathbf{x})_i > 0$. But then $(\mathbf{p}_0)_i = 0$ for those integers i between 1 and n for which $(\mathbf{x})_i > 0$, and therefore $\mathbf{p}_0 \,.\, \mathbf{x} = 0$. We conclude from this that if $(\mathbf{p}, 0) \in N$ and $\mathbf{x} \in B_{\mathbf{c}}(\mathbf{p}, 0)$ then $\mathbf{p}_0 \,.\, \mathbf{x} = 0$, and therefore $\mathbf{x} \in B_{\mathbf{c}}(\mathbf{p}_0, 0)$. But

$$B_{\mathbf{c}}(\mathbf{p}_0, 0) \subset B_{\mathbf{c}}(\mathbf{p}_0, w_0) \subset V.$$

We conclude therefore that if $(\mathbf{p}, 0) \in N$ then $B_{\mathbf{c}}(\mathbf{p}, 0) \subset V$.

Now let $(\mathbf{p}, w) \in N$, where $w > 0$, and let $\mathbf{x} \in B_{\mathbf{c}}(\mathbf{p}, w)$. Then $\mathbf{x} \geq 0$ and $\mathbf{p} \,.\, \mathbf{x} \leq w$. Then

$$\mathbf{p}_0 \,.\, \mathbf{x} = \sum_{i=1}^{n} (\mathbf{p}_0)_i (\mathbf{x})_i \leq \frac{w_1}{w} \sum_{i=1}^{n} (\mathbf{p})_i (\mathbf{x})_i = \frac{w_1}{w} \mathbf{p} \,.\, \mathbf{x} \leq w_1,$$

and therefore $\mathbf{x} \in B_{\mathbf{c}}(\mathbf{p}_0, w_1)$. It follows that if $(\mathbf{p}, w) \in N$ and $w > 0$ then

$$B_{\mathbf{c}}(\mathbf{p}, w) \subset B_{\mathbf{c}}(\mathbf{p}_0, w_1) \subset V.$$

We conclude therefore that $B_{\mathbf{c}}(\mathbf{p}, w) \subset V$ for all $(\mathbf{p}, w) \in N$. The results we have so far obtained combine to show that the correspondence $B_{\mathbf{c}}$ is upper hemicontinuous on $\mathbb{R}^n_+ \times \mathbb{R}_+$.

Now let $(\mathbf{p}_0, w_0) \in \mathbb{R}^n_+ \times \mathbb{R}_+$ satisfy $w_0 > 0$, and let V be an open set in $\mathbb{R}^n$ that satisfies $V \cap B_{\mathbf{c}}(\mathbf{p}_0, w_0) \neq \emptyset$. The constraint $w_0 > 0$ ensures that any open ball of positive radius centred on a point of $B_{\mathbf{c}}(\mathbf{p}_0, w_0)$ intersects the interior of that set. It follows that the open set V must intersect the interior of the set $B_{\mathbf{c}}(\mathbf{p}_0, w_0)$, and therefore there exists $\mathbf{x}_0 \in V$ for which $\mathbf{0} \leq \mathbf{x}_0 \leq \mathbf{c}$ and $\mathbf{p}_0 \,.\, \mathbf{x}_0 < w_0$. Let

$$N = \{(\mathbf{p}, w) \in \mathbb{R}^n_+ \times \mathbb{R}_+ : w - \mathbf{p} \,.\, \mathbf{x}_0 > 0\}.$$

Then N is open in $\mathbb{R}^n$, $(\mathbf{p}_0, \mathbf{w}_0) \in N$, and $\mathbf{x}_0 \in V \cap B_{\mathbf{c}}(\mathbf{p}, w)$ for all $(\mathbf{p}, w) \in N$. We conclude from this that the correspondence $B_{\mathbf{c}}$ is lower hemicontinuous on the set $\mathbb{R}^n_+ \times \mathbb{R}_+$. This completes the proof. $\blacksquare$