MA3486—Fixed Point Theorems and Economic Equilibria
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7.3. Quasiconvex Functions

Definition

Let K be a convex set in some real vector space. A real-valued function $f\colon K \to \mathbb{R}$ is said to be *quasiconvex* if

$$f((1-t)\mathbf{u} + t\mathbf{v}) \leq \max\Big(f(\mathbf{u}), f(\mathbf{v})\Big)$$

for all $\mathbf{u}, \mathbf{v} \in K$ and for all real numbers t satisfying $0 \leq t \leq 1$.

Definition

Let K be a convex set in some real vector space. A real-valued function $f\colon K \to \mathbb{R}$ is said to be *quasiconcave* if

$$f((1-t)\mathbf{u} + t\mathbf{v}) \geq \min\Big(f(\mathbf{u}), f(\mathbf{v})\Big)$$

for all $\mathbf{u}, \mathbf{v} \in K$ and for all real numbers t satisfying $0 \leq t \leq 1$.

Linear functionals are quasiconvex and quasiconcave.

A function $f\colon K \to \mathbb{R}$ defined over a compact subset K of a real vector space is quasiconcave if and only if the function $-f$ is quasiconvex.

Lemma 7.2

Let K be a convex set in a real vector space, and let $f: K \to \mathbb{R}$ be a quasiconcave function. Then, for each real number s, the preimage $f^{-1}([s, +\infty))$ of the interval $[s, +\infty)$ is a convex subset of K, where

$$f^{-1}([s, +\infty)) = \{\mathbf{x} \in K : f(\mathbf{x}) \geq s\}.$$

Proof

Let $\mathbf{u}, \mathbf{v} \in f^{-1}([s, +\infty))$, and let t be a real number satisfying $0 \leq t \leq 1$. Then $f(\mathbf{u}) \geq s$ and $f(\mathbf{v}) \geq s$. It follows from the definition of quasiconcavity that

$$f((1-t)\mathbf{u} + t\mathbf{v}) \geq \min\Big(f(\mathbf{u}), f(\mathbf{v})\Big) \geq s,$$

and therefore $(1-t)\mathbf{u} + t\mathbf{v} \in f^{-1}([s, +\infty))$, as required. ∎

7.4. Nash Equilibria

We consider a *game* with n players. Each player choses a strategy from an appropriate *strategy sets*. The strategies chosen by the players in the game constitute a *strategy profile*. The *utility*, or *payoff*, of the game, for each player is determined by the strategy profile chosen by the players in the game. The technical details involved are explored and specified in more detail in the following discussion.

We suppose that, in an n-player game, the ith player choses strategies from a *strategy set* S_i, where S_i is represented as a non-empty compact convex set in $\mathbb{R}^{m_i}$ for some positive integer m_i. (The convexity requirement would typically be satisfied in games where players can adopt mixed strategies.) We let $S = S_1 \times S_1 \times \cdot \times S_n$. The elements of the set S are referred to as *strategy profiles*. The *strategy profile set* S is a compact convex subset of $\mathbb{R}^m$, where

$$m = m_1 + m_2 + \cdots + m_n.$$

For each integer i between 1 and n let us define

$$\begin{array}{rcl} S_{-1} & = & S_2 \times S_3 \times S_4 \times \cdots \times S_n, \\ S_{-2} & = & S_1 \times S_3 \times S_4 \times \cdots \times S_n, \\ S_{-3} & = & S_1 \times S_2 \times S_4 \times \cdots \times S_n, \\ & \vdots & \\ S_{-n} & = & S_1 \times S_2 \times S_3 \times \cdots \times S_{n-1}, \end{array}$$

so that

$$S_{-i} = S_1 \times \cdots \times S_{i-1} \times S_{i+1} \times \cdots \times S_n$$

for all integers i between 1 and n (making the appropriate interpretation of the right hand side of this expression, as specified above, in the cases $i = 1$ and $i = n$). The set S_{-i} is then a compact convex subset of $\mathbb{R}^{m-m_i}$ for $i = 1, 2, \ldots, n$.

We define projections $\pi_i \colon S \to S_i$ and $\pi_{-i} \colon S \to S_{-i}$ for $i = 1, 2, \ldots, n$ in the obvious fashion so that

$$\pi_i(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n) = \mathbf{x}_i$$

and

$$\begin{array}{rcl}
\pi_{-1}(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n) & = & (\mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4, \ldots, \mathbf{x}_n), \\
\pi_{-2}(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n) & = & (\mathbf{x}_1, \mathbf{x}_3, \mathbf{x}_4, \ldots, \mathbf{x}_n), \\
\pi_{-3}(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n) & = & (\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_4, \ldots, \mathbf{x}_n), \\
 & \vdots & \\
\pi_{-n}(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n) & = & (\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots, \mathbf{x}_{n-1}).
\end{array}$$

We now consider the utility, or payoff, of the game for the players. We suppose that, for each integer i between 1 and n, the *utility* of the game, from the perspective of the ith player, is determined by a utility function $u_i \colon S_i \times S_{-i} \to \mathbb{R}$ defined so that, for each element $\mathbf{x}_{-i}$ of S_{-i} representing a choice of strategies by players of the game other than the ith player, the real number $u_i(\mathbf{x}_i, \mathbf{x}_{-i})$ represents the utility, or payoff, for the ith player on adopting the strategy $\mathbf{i}$. We impose the following two requirements on these utility functions:

- the utility function $u_i \colon S_i \times S_{-i} \to \mathbb{R}$ is continuous on $S_i \times S_{-i}$;
- for fixed $\mathbf{x}_{-i}$, the function sending $\mathbf{x}_i$ to $u_i(\mathbf{x}_i, \mathbf{x}_{-i})$ is quasiconcave on S_i, and thus

$$u_i((1-t)\mathbf{x}_i' + t\mathbf{x}_i'', \mathbf{x}_{-i}) \geq \min\Bigl(u_i(\mathbf{x}_i', \mathbf{x}_{-i}), u_i(\mathbf{x}_i'', \mathbf{x}_{-i})\Bigr)$$

  for all $\mathbf{x}_i', \mathbf{x}_i'' \in S_i$, $\mathbf{x}_{-i} \in S_{-i}$ and real numbers t satisfying $0 \leq t \leq 1$.

Let $\mathbf{x}'_i$ and $\mathbf{x}''_i$ elements of the strategy set S_i, representing strategies for the ith player, and let $\mathbf{x}_{-i}$ be an element of S_{-i}, representing a profile of strategies adopted by the other players. Then the ith player actively prefers the outcome of strategy profile $\mathbf{x}''_i$ to that of strategy profile $\mathbf{x}'_i$ if and only if

$$u_i(\mathbf{x}'_i, \mathbf{x}_{-i}) < u_i(\mathbf{x}''_i, \mathbf{x}_{-i}).$$

The ith player is indifferent between the outcomes of the strategy profiles $\mathbf{x}'_i$ and $\mathbf{x}''_i$ if and only if

$$u_i(\mathbf{x}'_i, \mathbf{x}_{-i}) = u_i(\mathbf{x}''_i, \mathbf{x}_{-i}).$$

Definition

In an n-player game, let $S_1, S_2, \ldots, S_n$ denote the strategy sets for the players in the game, and let $u_i \colon S_i \times S_{-i} \to \mathbb{R}$ denote the utility function for the ith player in the game (where the set S_{-i} is defined for $i = 1, 2, \ldots, n$ as described above). A strategy profile

$$(\mathbf{x}_1^*, \mathbf{x}_2^*, \ldots, \mathbf{x}_n^*)$$

is said to be a *Nash equilibrium* for the game if

$$u_i(\mathbf{x}_i, \mathbf{x}_{-i}^*) \le u_i(\mathbf{x}_i^*, \mathbf{x}_{-i}^*).$$

for all integers i between 1 and n and for all $\mathbf{x}_i \in S_i$.

Given any element $\mathbf{x}_{-i}$ of S_{-i} (representing a choice of strategies that might be adopted by the other players of the game), there will be a subset $B_i(\mathbf{x}_{-i})$ of S_i that represents the best strategies that the ith player can adopt when the other players are adopting the strategies represented by the element $\mathbf{x}_{-i}$ of S_{-i}. These best strategies are those strategies that maximize the utility function for the ith player, and we denote the value of the utility function u_i for those best strategies by $b_i(\mathbf{x}_{-i})$. Accordingly

$$
\begin{aligned}
b_i(\mathbf{x}_{-i}) &= \sup\{u_i(\mathbf{x}_i, \mathbf{x}_{-i}) : \mathbf{x}_i \in S_i\}, \\
B_i(\mathbf{x}_{-i}) &= \{\mathbf{x}_i \in S_i : u_i(\mathbf{x}_i, \mathbf{x}_{-i}) = b(\mathbf{x}_{-i})\}.
\end{aligned}
$$

We obtain in this fashion a single-valued function $b_i \colon S_{-i} \to S_i$ and a correspondence $B_i \colon S_{-i} \rightrightarrows S_i$.

Now, for each integer i between 1 and n, the constant correspondence that sends each element of S_{-i} to the strategy set S_i is clearly both upper hemicontinuous and lower hemicontinuous. The function $u_i \colon S_i \times S_{-i} \to \mathbb{R}$ is required to be continuous. Moreover, for each $\mathbf{x}_{i-1} \in S_{-i}$, the Extreme Value Theorem ensures that the set $B_i(\mathbf{x}_{-i})$ is non-empty, and the continuity of the utility function u_i ensures that $B_i(\mathbf{x}_{-i})$ is a closed subset of the compact set S_i. It follows that the the correspondence $B \colon S_{-i} \rightrightarrows S_i$ is both non-empty and compact. It therefore follows from Berge's Maximum Theorem (Theorem 2.23) that the function $b \colon S_{-i} \to \mathbb{R}$ is continuous on S_{-i}, $B_i(\mathbf{x}_{-i})$ is a compact subset of S_i for all $\mathbf{x}_{-i} \in S_{-i}$, and the correspondence $B \colon S_{-i} \rightrightarrows S_i$ is upper hemicontinuous in S_{-i}. Now every upper hemicontinuous closed-valued correspondence has a closed graph (Proposition 2.11). We conclude therefore that the correspondence $B \colon S_{-i} \rightrightarrows S_i$ has a closed graph.

Now, for each i, and for each $\mathbf{x}_{-i} \in S_{-i}$, the quasiconcavity requirement imposed on the utility function i ensures that the non-empty compact set $B_i(\mathbf{x}_{-i})$ is convex. Indeed the definition of $b_i(\mathbf{x}_{-i})$ and $B_i(\mathbf{x}_{-i})$ ensures that $u_i(\mathbf{z}, \mathbf{x}_{-i}) \leq b_i(\mathbf{x}_{-i})$ for all $\mathbf{z} \in S_i$, and $u_i(\mathbf{z}, \mathbf{x}_{-i}) = b_i(\mathbf{x}_{-i})$ for all $\mathbf{z} \in B_i(\mathbf{x}_{-i})$. It follows that

$$B_i(\mathbf{x}_{-i}) = \{\mathbf{z} \in S_i : u_i(\mathbf{z}, \mathbf{x}_{-i}) \geq b(\mathbf{x}_{-i})\}.$$

The quasiconcavity condition on the function u_i ensures that, for all $\mathbf{z}, \mathbf{z}' \in B_i(\mathbf{x}_{-i})$ and for all real numbers t satisfing $0 \leq t \leq 1$,

$$u_i((1-t)\mathbf{z}' + t\mathbf{z}'', \mathbf{x}_{-i}) \geq \min\Big(u_i(\mathbf{z}', \mathbf{x}_{-i}), u_i(\mathbf{z}'', \mathbf{x}_{-i})\Big) \geq b(\mathbf{x}_{-i}),$$

and therefore $(1-t)\mathbf{z}' + t\mathbf{z}'' \in B_i(\mathbf{x}_{-i})$. (This justification of the convexity of $B_i(\mathbf{x}_{-i})$ essentially repeats the argument presented in the proof of Lemma 7.2.)

We have now shown that, for each integer i between 1 and n, the correspondence $B_i \colon S_{-i} \to S_i$ that assigns to each element $\mathbf{x}_{-i}$ of S_{-i} the set of best strategies that the ith player can adopt in the event that the other players adopt the strategies represented by $\mathbf{x}_{-i}$ has closed graph, and maps each element $\mathbf{x}_{-i}$ of S_{-i} to a subset $B_i(\mathbf{x}_{-i})$ that is non-empty, compact and convex.

Now the Kakutani Fixed Point Theorem (Theorem 5.4) applies to correspondences with closed graph that map elements of a non-empty, compact and convex subset to non-empty convex subsets of that set. Thus in order to obtain a proof of the existence of Nash equilibria that utilizes the Kakutani Fixed Point Theorem, we must construct such a correspondence from a non-empty compact convex set to itself.

We recall that the *strategy profile set* S is defined to be the Cartesian product $S_1 \times S_2 \times \cdots \times S_n$ of the strategy sets for the players of the game. Let $\Phi\colon S \rightrightarrows S$ be the correspondence from the strategy profile set S to itself defined so that

$$\Phi(\mathbf{x}) = \Big(B_1(\pi_{-1}(\mathbf{x})), B_2(\pi_{-2}(\mathbf{x})), \cdots B_n(\pi_{-n}(\mathbf{x})) \Big)$$

for $i = 1, 2, \ldots, n$. Then

$$\{(\mathbf{x}, \mathbf{y}) \in S \times S : \mathbf{y} \in \Phi(\mathbf{x})\} = \bigcap_{i=1}^{n} G_i,$$

where

$$G_i = \{(\mathbf{x}, \mathbf{y}) \in S \times S : \pi_i(\mathbf{y}) \in B_i(\pi_{-i}(\mathbf{x}))\}$$

for $i = 1, 2, \ldots, n$.

Now, for each i, the set

$$\{(\mathbf{x}_{-i}, \mathbf{y}_i) \in S_{-i} \times S_i : \mathbf{y}_i \in B_i(\mathbf{x}_{-i})\}$$

is closed in $S_{-i} \times S_i$, because the correspondence $B_i \colon S_{-i} \rightrightarrows S_i$ has closed graph. It follows that each set G_i is closed in $S \times S$, because the set G_i is the preimage of a closed set under the continuous mapping from $S \times S$ to $S_{-i} \times S_i$ that sends each ordered pair $(\mathbf{x}, \mathbf{y})$ in $S \times S$ to $(\pi_{-i}(\mathbf{x}), \pi_i(\mathbf{y}))$. The graph of the correspondence Φ is the intersection of the closed sets $G_1, G_2, \ldots, G_n$. It is therefore itself a closed set. Thus the correspondence $\Phi \colon S \rightrightarrows S$ has closed graph. Moreover S is a non-empty compact convex set, and $\Phi(\mathbf{x})$ is a non-empty convex subset of S for all $\mathbf{x} \in S$. It follows from the Kakutani Fixed Point Theorem (Theorem 5.4) that there exists a fixed point $\mathbf{x}^*$ for the correspondence Φ. This fixed point is strategy profile that satisfies $\mathbf{x}^* \in \Phi(\mathbf{x}^*)$.

Let $\mathbf{x}_i^* = \pi_i(\mathbf{x}^*)$ and $\mathbf{x}_{-i}^* = \pi_{-i}(\mathbf{x}^*)$ for $i = 1, 2, \ldots, n$. Then $\mathbf{x}_i^* \in B_i(\mathbf{x}_{-i}^*)$ for $i = 1, 2, \ldots, n$, because $\mathbf{x}^* \in \Phi(\mathbf{x}^*)$. It follows from the definition of $B_i(\mathbf{x}_{-i}^*$ that

$$u_i(\mathbf{x}_i, \mathbf{x}_{-i}^*) \leq u_i(\mathbf{x}_i^*, \mathbf{x}_{-i}^*)$$

for all integers i between 1 and n and for all $\mathbf{x}_i \in S_i$. The strategy profile $(\mathbf{x}_1^*, \mathbf{x}_2^*, \ldots, \mathbf{x}_n^*)$ therefore represents a Nash equilibrium for the game.

Theorem 7.3 (Existence of Nash Equilibria)

Consider an n-person game in which, for each integer i between 1 and n, the strategy set S_i is a compact convex subset of a Euclidean space, and in which the utility function $u_i\colon S_i \times S_{i-1} \to \mathbb{R}$ that determines the utility for the ith player, given a strategy profile $\mathbf{x}_{-i}$ representing strategies chosen by the other players, is a continuous function that, for any fixed $\mathbf{x}_{-i} \in S_{-i}$, determines a quasiconcave function mapping $\mathbf{x}_i$ to $u_i(\mathbf{x}_i, \mathbf{x}_{-i})$ as $\mathbf{x}_i$ varies over the strategy set S_i. Then there exists a Nash equilibrium $(\mathbf{x}_1^*, \mathbf{x}_2^*, \ldots, \mathbf{x}_n^*)$ for the game. Accordingly

$$u_i(\mathbf{x}_i, \mathbf{x}_{-i}^*) \leq u_i(\mathbf{x}_i^*, \mathbf{x}_{-i}^*)$$

for all integers i between 1 and n and for all $\mathbf{x}_i \in S_i$.