MA3486—Fixed Point Theorems and Economic Equilibria
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7. Game Theory and Nash Equilibria

7.1. Zero-Sum Two-Person Games

Example

Consider the following hand game. This is a zero-sum two-person game. At each go, the two players present simultaneously either and open hand or a fist. If both players present fists, or if both players present open hands, then no money changes hands. If one player presents a fist and the other player presents an open hand then the player presenting the fist receives ten cents from the player presenting the open hand.

The payoff for the first player can be represented by the following payoff matrix:

$$\begin{pmatrix} 0 & -10 \\ 10 & 0 \end{pmatrix}.$$

In this matrix the entry in the first row represent the payoffs when the first player presents an open hand; those in the second row represent the payoffs when the first player presents a fist. The entries in the first column represent the payoff when the second player presents an open hand; those in the second column represent the payoffs when the second player presents a fist. In this game the second player, choosing the best strategy, is always going to play a fist, because that reduces the payoff for the first player, whatever the first player chooses to play. Similarly the first player, choosing the best strategy, is going to play a fist, because that maximizes the payoff for the first player whatever the second player does. Thus in this game, both players choosing the best strategies, play fists.

It should be noticed that, in this situation, if the second player always plays a fist, the first player would not be tempted to move from a strategy of always playing a fist in order get a better payoff. Similarly if the first player always plays a fist, then the second player would not be tempted to move from a strategy of always playing a fist in order to reduce the payoff to the first player. This is a very simple example of a *Nash Equilibrium*. This equilibrium arises because the element in the second row and second column of the payoff matrix is simultaneously the largest element in its column and the smallest element in its row. Matrix elements with this property as said to be *saddle points* of the matrix.

Example

Now consider the game of *Rock, Paper, Scissors*. This game has a long history, and versions of this game were well-established in China and Japan in particular for many centuries.

Two players simultaneously present hand symbols representing *Rock* (a closed fist), *Paper* (a flat hand), or *Scissors* (first two fingers outstretched in a 'V'). Paper beats Rock, Scissors beats Paper, Rock beats Scissors. If both players present the same hand symbol then that round is a draw.

Ordering the strategies for the playes in the order *Rock* (1st), *Paper* (2nd) and *Scissors* (3rd), the payoff matrix for the first player is the following:—

$$\begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}.$$

The entry in the ith row and jth column of this payoff matrix represents the return to the first player on a round of the game if the first player plays strategy i and the second player plays strategy j.

A *pure strategy* would be one in which a player presents the same hand symbol in every round. But it is not profitable for any player in this game to adopt a pure strategy. If the first player adopts a strategy of playing *Paper*, then the second player, on observing this, would adopt a strategy of always playing *Scissors*, and would beat the first player on every round. A preferable strategy, for each player, is the *mixed strategy* of playing *Rock*, *Paper* and *Scissors* with equal probability, and seeking to ensure that the sequence of plays is as random as possible.

Let us denote by M the payoff matrix above. A *mixed strategy* for the first player is one in which, on any given round *Rock* is played with probability p_1, *Paper* is played with probability p_2 and *Scissors* is played with probability p_3. The mixed strategies for the first player can therefore be represented by points of a triangle Δ_P, where

$$\begin{aligned}\Delta_P &= \Big\{(p_1, p_2, p_3) \in \mathbb{R}^3 : \\ &\qquad p_1 \geq 0,\;\; p_2 \geq 0,\;\; p_3 \geq 0,\;\; p_1 + p_2 + p_3 = 1\Big\}.\end{aligned}$$

A mixed strategy for the second player is one in which *Rock* is played with probability q_1, *Paper* with probability q_2 and *Scissors* with probability q_3. The mixed strategies for the second player can therefore be represented by points of a triangle Δ_Q, where

$$\begin{aligned}\Delta_Q &= \Big\{(q_1, q_2, q_3) \in \mathbb{R}^3 : \\ &\qquad q_1 \geq 0, \;\; q_2 \geq 0, \;\; q_3 \geq 0, \;\; q_1 + q_2 + q_3 = 1\Big\}.\end{aligned}$$

Let $\mathbf{p} \in \Delta_P$ represent the mixed strategy chosen by the first player, and $\mathbf{q} \in \Delta_Q$ the mixed strategy chosen by the second player, where

$$\mathbf{p} = (p_1, p_2, p_3), \quad \mathbf{q} = (q_1, q_2, q_3).$$

Let M_{ij} the payoff for the first player when the first player plays strategy i and the second player plays strategy j. Then M_{ij} is the entry in the ith row and jth column of the payoff matrix M. In matrix equations we consider $\mathbf{p}$ and $\mathbf{q}$ to be column vectors, denoting their transposes by the row matrices $\mathbf{p}^T$ and $\mathbf{q}^T$. The *expected payoff* for the first player is then $f(\mathbf{p}, \mathbf{q})$, where

$$f(\mathbf{p}, \mathbf{q}) = \mathbf{p}^T M\mathbf{q} = \sum_{i=1}^{3} \sum_{j=1}^{3} p_i M_{ij} q_j.$$

Let $\mathbf{p}^* = (p_1^*, p_2^*, p_3^*)$, where

$$p_1^* = p_2^* = p_3^* = \tfrac{1}{3}.$$

Then ${\mathbf{p}^*}^T M = (0, 0, 0)$, and therefore

$$f(\mathbf{p}^*, \mathbf{q}) = 0$$

for all $\mathbf{q} \in \Delta_Q$. Similarly let $\mathbf{q}^* = (q_1^*, q_2^*, q_3^*)$, where

$$q_1^* = q_2^* = q_3^* = \tfrac{1}{3}.$$

Then

$$f(\mathbf{p}, \mathbf{q}^*) = 0$$

for all $\mathbf{p} \in \Delta_Q$. Thus the inequalities

$$f(\mathbf{p}, \mathbf{q}^*) \leq f(\mathbf{p}^*, \mathbf{q}^*) \leq f(\mathbf{p}^*, \mathbf{q})$$

are satisfied for all $\mathbf{p} \in \Delta_P$ and $\mathbf{q} \in \Delta_q$, because each of the quantities occurring is equal to zero.

Were the first player to adopt a mixed strategy $\mathbf{p}$, where $\mathbf{p} = (p_1, p_2, p_3)$, $p_i \geq 0$ for $i = 1, 2, 3$ and $p_1 + p_2 + p_3 = 1$, the second player could adopt mixed strategy $\mathbf{q}$, where $\mathbf{q} = (q_1, q_2, q_3) = (p_3, p_1, p_2)$. The payoff $f(\mathbf{p}, \mathbf{q})$ is then

$$
\begin{aligned}
f(\mathbf{p}, \mathbf{q}) &= -p_1 q_2 + p_1 q_3 - p_2 q_3 + p_2 q_1 - p_3 q_1 + p_3 q_2 \\
&= -p_1^2 + p_1 p_2 - p_2^2 + p_2 p_3 - p_3^2 + p_3 p_1 \\
&= -\frac{1}{6}\Big((2p_1 - p_2 - p_3)^2 + (2p_2 - p_3 - p_1)^2 \\
&\qquad + (2p_3 - p_1 - p_2)^2\Big) \\
&\leq 0.
\end{aligned}
$$

Moreover if $f(\mathbf{p}, \mathbf{q}) = 0$, where $q_1 = p_3$, $q_2 = p_1$ and $q_3 = p_2$, then

$$(2p_1 - p_2 - p_3)^2 + (2p_2 - p_3 - p_1)^2 + (2p_3 - p_1 - p_2)^2 = 0$$

and therefore $2p_1 = p_2 + p_3$, $2p_2 = p_3 + p_1$ and $2p_3 = p_1 + p_2$. But then

$$3p_1 = 3p_2 = 3p_3 = p_1 + p_2 + p_3 = 1,$$

and thus $\mathbf{p} = \mathbf{p}^*$. It follows that if $\mathbf{p} \in \Delta_Q$ and $\mathbf{p} \neq \mathbf{p}^*$ then there exists $\mathbf{q} \in \Delta_Q$ for which $f(\mathbf{p}, \mathbf{q}) < 0$. Thus if the first player adopts a mixed strategy other than the strategy $\mathbf{p}^*$ in which *Rock*, *Paper*, *Scissors* are played with equal probability on each round, there is a mixed strategy for the second player that ensures that the average payoff for the first player is negative, and thus the first player will lose in the long run over many rounds. Thus strategy $\mathbf{p}^*$ is the only sensible mixed strategy that the first player can adopt. The corresponding strategy $\mathbf{q}^*$ is the only sensible mixed strategy that the second player can adopt. The average payoff for each player is then equal to zero.

7.2. Von Neumann's Minimax Theorem

In 1920, John Von Neumann published a paper entitled "Zur Theorie der Gesellschaftsspielle" (*Mathematische Annalen*, Vol. 100 (1928), pp. 295–320). The title translates as "On the Theory of Social Games". This paper included a proof of the following "Minimax Theorem", which made use of the Brouwer Fixed Point Theorem. An alternative proof using results concerning convexity was presented in the book *On the Theory of Games and Economic Behaviour* by John Von Neumann and Oskar Morgenstern (Princeton University Press, 1944). George Dantzig, in a paper published in 1951, showed how the theorem could be solved using linear programming methods (see Joel N. Franklin, *Methods of Mathematical Economics*, (Springer Verlag, 1980, republished by SIAM in 1982).

Theorem 7.1 (Von Neumann's Minimax Theorem)

Let M be an $m \times n$ matrix, let

$$\begin{aligned}\Delta_P &= \Big\{(p_1, p_2, \ldots, p_m) \in \mathbb{R}^m : p_i \geq 0 \text{ for } i = 1, 2, \ldots, m, \text{ and} \\ &\qquad \sum_{i=1}^{m} p_i = 1\Big\}, \\ \Delta_Q &= \Big\{(q_1, q_2, \ldots, q_n) \in \mathbb{R}^n : q_i \geq 0 \text{ for } i = 1, 2, \ldots, n, \text{ and} \\ &\qquad \sum_{j=1}^{n} q_j = 1\Big\},\end{aligned}$$

and let

$$f(\mathbf{p}, \mathbf{q}) = \mathbf{p}^T M \mathbf{q} = \sum_{i=1}^{m} \sum_{j=1}^{n} M_{i,j} p_i q_j$$

for all $\mathbf{p} \in \Delta_P$ and $\mathbf{q} \in \Delta_Q$. Then there exist $\mathbf{p}^* \in \Delta_P$ and $\mathbf{q}^* \in \Delta_Q$ such that

$$f(\mathbf{p}, \mathbf{q}^*) \leq f(\mathbf{p}^*, \mathbf{q}^*) \leq f(\mathbf{p}^*, \mathbf{q})$$

for all $\mathbf{p} \in \Delta_P$ and $\mathbf{q} \in \Delta_Q$.

Proof

Let $f(\mathbf{p},\mathbf{q}) = \mathbf{p}^T M\mathbf{q}$ for all $\mathbf{p} \in \Delta_P$ and $\mathbf{q} \in \Delta_Q$. Given $\mathbf{q} \in \Delta_Q$, let

$$\mu_P(\mathbf{q}) = \sup\{f(\mathbf{p},\mathbf{q}) : \mathbf{p} \in \Delta_P\}$$

and let

$$P(\mathbf{q}) = \{\mathbf{p} \in \Delta_P : f(\mathbf{p},\mathbf{q}) = \mu_P(\mathbf{q})\}.$$

Similarly given $\mathbf{p} \in \Delta_P$, let

$$\mu_Q(\mathbf{p}) = \inf\{f(\mathbf{p},\mathbf{q}) : \mathbf{q} \in \Delta_Q\}$$

and let

$$Q(\mathbf{p}) = \{\mathbf{q} \in \Delta_Q : f(\mathbf{p},\mathbf{q}) = \mu_Q(\mathbf{q})\}.$$

An application of Berge's Maximum Theorem (Theorem 2.23) ensures that the functions $\mu_P\colon \Delta_P \to \mathbb{R}$ and $\mu_Q\colon \Delta_Q \to \mathbb{R}$ are continuous, and that the correspondences $P\colon \Delta_Q \rightrightarrows \Delta_P$ and $Q\colon \Delta_P \rightrightarrows \Delta_Q$ are non-empty, compact-valued and upper hemicontinuous. These correspondences therefore have closed graphs (see Proposition 2.11). Morever $P(\mathbf{q})$ is convex for all $\mathbf{q} \in \Delta_Q$ and $Q(\mathbf{p})$ is convex for all $\mathbf{p} \in \Delta_P$. Let $X = \Delta_P \times \Delta_Q$, and let $\Phi\colon X \rightrightarrows X$ be defined such that

$$\Phi(\mathbf{p}, \mathbf{q}) = P(\mathbf{q}) \times Q(\mathbf{p})$$

for all $(\mathbf{p}, \mathbf{q}) \in X$. Kakutani's Fixed Point Theorem (Theorem 5.4) then ensures that there exists $(\mathbf{p}^*, \mathbf{q}^*) \in X$ such that $(\mathbf{p}^*, \mathbf{q}^*) \in \Phi(\mathbf{p}^*, \mathbf{q}^*)$. Then $\mathbf{p}^* \in P(\mathbf{q}^*)$ and $\mathbf{q}^* \in Q(\mathbf{p}^*)$ and therefore

$$f(\mathbf{p}, \mathbf{q}^*) \leq f(\mathbf{p}^*, \mathbf{q}^*) \leq f(\mathbf{p}^*, \mathbf{q})$$

for all $\mathbf{p} \in \Delta_P$ and $\mathbf{q} \in \Delta_Q$, as required. ∎