MA3486—Fixed Point Theorems and Economic Equilibria
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Proposition 6.8

Let T be a non-negative $n \times n$ (square) matrix, and let μ denote the Perron root of T. Let I denote the identity $n \times n$ matrix. Then, given any σ is a non-negative real number satisfying $\mu\sigma < 1$, the matrix $I - \sigma T$ is invertible and $(1 - \sigma T)^{-1}$ is a non-negative matrix.

Proof

We use some basic results of linear algebra and complex analysis. Let z be a complex number. Then the eigenvectors of the matrix $I - zT$ are the same as those of the matrix T, and therefore the eigenvalues of $I - zT$ are of the form $1 - z\lambda$ as λ ranges of the eigenvales of T.

Now the modulus of any eigenvalue of the non-negative matrix T is bounded above by the Perron root of T (Proposition 6.7). Therefore the eigenvalues of $I - zT$ have real part not less than $1 - |z|\mu$. A square matrix is invertible if zero is not an eigenvalue of that matrix. It follows that the matrix $I - zT$ is invertible for all complex numbers z satisfying $\mu|z| < 1$.

The determinant of the matrix $I - zT$ is a polynomial function of z. It follows that if $\mu > 0$ then all coefficients of the matrix $(I - zT)^{-1}$ are holomorphic functions of the complex variable z throughout the disk $\{z \in \mathbb{C} : |z| < \mu^{-1}\}$, and if $\mu = 0$ then all coefficients of the matrix $(I - zT)^{-1}$ are holomorphic functions of the complex variable z throughout entire complex plane. A basic theorem of complex analysis therefore ensures that each coefficient of the matrix $(I - zT)^{-1}$ may be represented as a power series in the complex plane z that converges for all complex numbers z satisfying $\mu|z| < 1$.

Now

$$(1 - zT)(1 + zT + z^2T^2 + z^3T^3 + \cdots + z^kT^k) = 1 - z^{k+1}T^{k+1},$$

and thus

$$\begin{aligned}(1 - zT)^{-1} &= 1 + zT + z^2T^2 + z^3T^3 + \cdots + z^kT^k \\ &\quad + z^{k+1}T^{k+1}(I - zT)^{-1}\end{aligned}$$

when $\mu|z| < 1$.

Now it has already been shown that $(1 - zT)^{-1}$ can be represented by a power series in z that converges whenever $\mu|z| < 1$. we can therefore conclude that

$$(1 - zT)^{-1} = 1 + zT + z^2T^2 + z^3T^3 + \cdots$$

for all complex numbers z satisfying $\mu|z| < 1$.

In particular

$$(1 - \sigma T)^{-1} = 1 + \sigma T + \sigma^2 T^2 + \sigma^3 T^3 + \cdots$$

for all non-negative real numbers σ satisfying $\mu\sigma < 1$. But each summand on the right side of this power series representation of $(1 - \sigma T)^{-1}$ is a non-negative matrix. It follows that $I - \sigma T$ is invertible and $(1 - \sigma T)^{-1}$ is a non-negative matrix for all non-negative real numbers σ satisfying $\sigma\rho < 1$, as required. ∎

Proposition 6.9

Let T be a non-negative $n \times n$ (square) matrix, let μ denote the Perron root of T. Then the Perron root of the transpose T^T is equal to the Perron root μ of T, and there exists a non-zero vector $\mathbf{p} \in \mathbb{R}^n$ satisfying $\mathbf{p} \geq \mathbf{0}$ and $\mathbf{p}^T T = \mu \mathbf{p}^T$, where $\mathbf{p}^T$, the transpose of $\mathbf{p}$ is the row vector components are the components of the column vector $\mathbf{p}$.

Proof

The transpose T^T of the non-negative square matrix T is itself a non-negative square matrix with the same characteristic polynomial as T, and thus with the same eigenvalues as T. The Perron root of the transpose T^T of T is a non-negative real eigenvalue of T^T (Proposition 6.5), and moreover it is an upper bound on the modulus of every eigenvalue of T^T (Proposition 6.7. It follows that the non-negative square matrix T and its transpose T^T have the same Perron root. Moreover the Perron root is an eigenvalue of T^T, and therefore there exists a non-zero vector $\mathbf{p} \in \mathbb{R}^n$ for which $\mathbf{p} \geq 0$ and $T^T\mathbf{p} = \mu\mathbf{p}$. Taking the transpose of this equation, we find that $\mathbf{p}^T T = \mu\mathbf{p}^T$, as required. ∎

Proposition 6.10

Let T be a non-negative $n \times n$ (square) matrix, let μ denote the Perron root of T, and let σ is a non-negative real number. Then there exists a non-zero vector $\mathbf{w} \in \mathbb{R}^n$ satisfying $\mathbf{w} \geq \mathbf{0}$ and $\mathbf{w} >> \sigma T\mathbf{w}$ if and only if $\mu\sigma < 1$.

Proof

Let $\mathbf{v} \in \mathbb{R}^n$ satisfy $\mathbf{v} >> \mathbf{0}$, and let $\mathbf{w} = (I - \sigma T)^{-1}\mathbf{v}$, where I denotes the identity $n \times n$ matrix. It follows from Proposition 6.8 that if $\mu\sigma < 1$ then $(I - \sigma T)^{-1}$ a non-negative matrix, and therefore $\mathbf{w} \geq 0$. Also

$$\mathbf{w} - \sigma T\mathbf{w} = (I - \sigma T)\mathbf{w} = \mathbf{v} >> \mathbf{0},$$

and therefore $\mathbf{w} >> \sigma T\mathbf{w}$. We have thus shown that if $\mu\sigma < 1$ then there exists a vector $\mathbf{w}$ with the required properties.

Conversely suppose that σ is a non-negative real number and that $\mathbf{w} \in \mathbb{R}^n$ is a non-zero vector for which $\mathbf{w} \geq 0$ and $\mathbf{w} >> \sigma T\mathbf{w}$. It follows from Proposition 6.9 that there exists a non-zero vector $\mathbf{p} \in \mathbb{R}^n$ satisfying $\mathbf{p} \geq \mathbf{0}$ and $\mathbf{p}^T T = \mu \mathbf{p}^T$, where $\mathbf{p}^T$ denotes the transpose of $\mathbf{p}$. Then

$$(1 - \sigma\mu)\mathbf{p}^T\mathbf{w} = \mathbf{p}^T\mathbf{w} - \sigma\mu\mathbf{p}^T\mathbf{w} = \mathbf{p}^T(\mathbf{w} - \sigma T\mathbf{w}) > 0.$$

It follows that $\mathbf{p}^T\mathbf{w} > 0$ and $\sigma\mu < 1$, as required. This completes the proof. ∎

6.2. Perron's Theorem for Positive Matrices

In 1907 Oskar Perron (1880–1975) proved a fundamental theorem concerning the eigenvalues and eigenvectors of a positive square matrix, in particular showing that the positive real number now referred to as the *Perron root* (or *Perron-Frobenius eigenvalue*) of the matrix is a simple eigenvector, with a one-dimensional eigenspace spanned by a positive eigenvector, and that any other eigenvalues of the matrix has a modulus strictly less than the Perron root. In 1912, Georg Frobenius (1849-1917) generalized Perron's Theorem to a particular class of non-negative square matrices that are said to be *unzerlegbar* (i.e., "indecomposible" or "irreducible"). These discoveries initiated the development of a body of results concerning non-negative square matrices that is today referred to as *Perron-Frobenius Theory*

Before stating and proving Perron's Theorem, we review (without proof) some standard results from linear algebra, related to the Jordan normal form of a square matrix, that are relevant to the proof of Perron's Theorem.

Let T be a linear operator defined on a finite-dimensional complex vector space V. Then the vector space V can be decomposed as a direct sum of subspaces that are invariant under the action of T and cannot be further decomposed as direct sums of invariant subspaces. Then

$$V = V_1 \oplus V_2 \oplus \cdots \oplus V_m$$

where, for each integer r between 1 and m, the linear operator T maps the subspace V_r of V into itself. Moreover the subspace V_r has no proper non-zero vector subspace that is invariant under the action of T. Associated with each subspace V_r is a complex number λ_r that is the unique eigenvalue of the restriction of the linear operator T to V_r.

The *characteristic polynomial* χ of T on V is defined such that $\chi(z) = \det(zI_V - T)$, where I_V denotes the identity operator on V. It can be shown that

$$\chi(z) = \prod_{r=1}^{m} (z - \lambda_r)^{d_r},$$

where $d_r = \dim_{\mathbb{C}} V_r$ for $r = 1, 2, \ldots, m$. It follows that a complex number λ is a simple root of the characteristic polynomial χ of T if and only if the following two conditions are satisfied: there exists exactly one integer r between 1 and m for which $\lambda = \lambda_r$; for this value of r, $d_r = 1$.

The theory of the Jordan Normal Form ensures that each subspace V_r has a basis of the form

$$\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_{d_r},$$

with the property that $T\mathbf{e}_1 = \lambda_r \mathbf{e}_s$ and $T\mathbf{e}_s = \lambda_r \mathbf{e}_s + \mathbf{e}_{s-1}$ for $1 < s \leq d_r$. All eigenvectors of T contained in V_r are scalar multiples of $\mathbf{e}_1$. Moreover if $d_r > 1$ then $(T - \lambda_r I_{V_r})^2 \mathbf{e}_2 = \mathbf{0}$ but $T\mathbf{e}_2 \neq \lambda_r \mathbf{e}_2$.

These results of linear algebra, summarized without detailed proof, yield the result stated in the following lemma.

Lemma 6.11

Let T be a linear operator acting on a finite-dimensional complex vector space V, and let λ be an eigenvalue of T. Then λ is a simple root of the characteristic polynomial of T if and only if the following two conditions are satisfied:

- *the eigenspace associated with the eigenvalue λ is one-dimensional;*
- *if $\mathbf{v} \in V$ satisfies the identity $(T - \lambda I_V)^2 \mathbf{v} = \mathbf{0}$ then $T\mathbf{v} = \lambda \mathbf{v}$.*

Theorem 6.12 (Perron)

Let T be a positive square matrix, and let μ be the Perron root of T. Then the following properties are satisfied:—

(i) *there exists an eigenvector of T with associated eigenvalue μ whose coefficients are all strictly positive;*

(ii) *the eigenvalue μ is a simple root of the characteristic polynomial of T, and the corresponding eigenspace is therefore one-dimensional;*

(iii) *all eigenvalues λ (real or complex) of T that are distinct from μ satisfy the inequality $|\lambda| < \mu$.*

Proof

Let the positive square matrix T be an $n \times n$ matrix, and let μ denote the Perron root of T. Proposition 6.4 establishes that the Perron root μ of T is well-defined and is an eigenvalue of T with which is associated an eigenvector $\mathbf{b}$ with positive coefficients. Moreover Proposition 6.4 ensures that the following properties are then satisfied:—

(iv) $\mathbf{b} >> \mathbf{0}$ and $T\mathbf{b} = \mu\mathbf{b}$;

(v) if ρ is a non-negative real number, if $\mathbf{v}$ is a non-zero n-dimensional vector with non-negative coefficients, and if $T\mathbf{v} \geq \rho\mathbf{v}$, then $\rho \leq \mu$.

(vi) given any n-dimensional vector $\mathbf{u}$ with real coefficients for which $T\mathbf{u} \geq \mu\mathbf{u}$, there exists a real number t for which $\mathbf{u} = t\mathbf{b}$, and thus $T\mathbf{u} = \mu\mathbf{u}$.

Now because the coefficients of the matrix T are all real, and μ is also a real number, the real and imaginary parts of any eigenvector of T with associated eigenvalue μ must themselves be eigenvectors with eigenvalue μ. The result just obtained therefore ensures that any convex eigenvector of T with eigenvalue μ must be a complex scalar multiple of the eigenvector $\mathbf{b}$. Thus the eigenspace of T associated with the eigenvalue μ is one-dimensional, when considered over the field of complex numbers.

Let I denote the identity $n \times n$ matrix, and let $\mathbf{v}$ be real n-dimensional vector for which $(T - \mu I)^2 \mathbf{v} = \mathbf{0}$. Then $T\mathbf{v} - \mu\mathbf{v}$ is an eigenvector of T with associated eigenvalue μ. It follows from property (vi) above that there must exist some real number α for which $T\mathbf{v} - \mu\mathbf{v} = \alpha\mathbf{b}$. Now $\mathbf{b} >> \mathbf{0}$. It follows that if $\alpha \geq 0$ then $T\mathbf{v} \geq \mu\mathbf{v}$. But property (vi) stated at the commencement of the proof then ensures that $\mathbf{v} = t\mathbf{b}$ for some real number t. But then $T\mathbf{v} = \mu\mathbf{v}$ and $\alpha = 0$. Similarly if $\alpha \leq 0$ then $T(-\mathbf{v}) \geq \mu(-\mathbf{v})$, and this also ensures that $\alpha = 0$. It follows that if $\mathbf{v}$ is a real n-dimensional vector satisfying $(T - \mu I)^2 \mathbf{v} = \mathbf{0}$ then $T\mathbf{v} = \mu\mathbf{v}$. The criterion stated in Lemma 6.11 therefore establishes that μ is a simple root of the characteristic polynomial of T.

We have now verified (i) and (ii). It remains to verify that all eigenvalues λ of T distinct from μ satisfy the inequality $|\lambda| < \mu$. Now it follows from Proposition 6.7 that all eigenvalues λ of T satisfy the inequality $|\lambda| \leq \mu$.

Now suppose that $|\lambda| = \mu$. It then follows from property (vi), stated at the commencement of the proof, that $T\mathbf{v} = \mu\mathbf{v} = |\lambda|\mathbf{v}$. It then follows from Lemma 6.6 that λ is a positive real number, and therefore $\lambda = \mu$. This completes the proof of (iii), and therefore completes the proof of the theorem. ∎