MA3486—Fixed Point Theorems and Economic Equilibria
School of Mathematics, Trinity College
Hilary Term 2018
Lecture 19 (March 8, 2018)

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6. Perron-Frobenius Theory

6.1. Eigenvectors of Non-Negative Matrices

We establish some notation that will be used throughout this section.

Let m and n be positive integers. Given any $m \times n$ matrix T, we denote by $(T)_{i,j}$ the coefficient in the ith row and jth column of the matrix T for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. Also given any n-dimensional vector $\mathbf{v}$, we denote by $(\mathbf{v})_j$ the jth coefficient of the vector j for $j = 1, 2, \ldots, n$.

Definition

A matrix T is said to be *non-negative* if all its coefficients are non-negative real numbers.

Definition

A matrix T is said to be *positive* if all its coefficients are strictly positive real numbers.

Let S and T be $m \times n$ matrices. If $(S)_{i,j} \leq (T)_{i,j}$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$, then we denote this fact by writing $S \leq T$, or by writing $T \geq S$. If $(S)_{i,j} < (T)_{i,j}$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$, then we denote this fact by writing $S << T$, or by writing $T >> S$.

Let $\mathbf{u}$ and $\mathbf{u}$ be n-dimensional vectors. If $(\mathbf{u})_j \leq (\mathbf{v})_j$ for $j = 1, 2, \ldots, n$, then we denote this fact by writing $\mathbf{u} \leq \mathbf{v}$, or by writing $\mathbf{v} \geq \mathbf{u}$. If $(\mathbf{u})_j < (\mathbf{v})_j$ for $j = 1, 2, \ldots, n$, then we denote this fact by writing $\mathbf{u} << \mathbf{v}$, or by writing $\mathbf{v} >> \mathbf{u}$.

A matrix T with real coefficients is thus *non-negative* if and only if $T \geq 0$. A matrix T with real coefficients is *positive* if and only if $T >> 0$.

Lemma 6.1

Let T be an $m \times n$ matrix with real coefficients. Then T is a non-negative matrix if and only if $T\mathbf{v} \geq 0$ for all $\mathbf{v} \in \mathbb{R}^n$ satisfying $\mathbf{v} \geq \mathbf{0}$.

Proof

Suppose that the matrix T is non-negative. Let $\mathbf{v} \in \mathbb{R}^n$ satisfy $\mathbf{v} \geq \mathbf{0}$. Then

$$(T\mathbf{v})_j = \sum_{k=1}^{n} (T)_{j,k}(\mathbf{v})_k \geq 0$$

for each integer j between 1 and m, because $(T)_{j,k}(\mathbf{v})_k \geq 0$ for $k = 1, 2, \ldots, n$. Therefore $T\mathbf{v} \geq \mathbf{0}$.

Conversely suppose that T is an $m \times n$ matrix with with real coefficients which has the property that if and only if $T\mathbf{v} \geq 0$ for all non-zero n-dimensional vectors $\mathbf{v}$ with non-negative real coefficients. Let $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ be the standard basis of $\mathbb{R}^n$ with

$$\mathbf{e}_1 = (1, 0, \ldots, 0), \quad \mathbf{e}_2 = (0, 1, \ldots, 0), \ldots, \quad \mathbf{0}_n = (0, 0, \ldots, 1).$$

Then $T\mathbf{e}_k \geq \mathbf{0}$ for $k = 1, 2, \ldots, n$, and therefore $(T)_{j,k} = (T\mathbf{e}_k)_j \geq 0$ for $j = 1, 2, \ldots, m$ and $k = 1, 2, \ldots, n$. The result follows. ∎

Lemma 6.2

Let T be an $m \times n$ matrix with real coefficients. Then T is a positive matrix if and only if $T\mathbf{v} >> \mathbf{0}$ for all $\mathbf{v} \in \mathbb{R}^n$ satisfying both $\mathbf{v} \neq \mathbf{0}$ and $\mathbf{v} \geq \mathbf{0}$.

Proof

Suppose that the matrix T is positive. Then $T_{j,k} > 0$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. Let $\mathbf{v} \in \mathbb{R}^n$ satisfy both $\mathbf{v} \neq \mathbf{0}$ and $\mathbf{v} \geq \mathbf{0}$. Then

$$(T\mathbf{v})_j = \sum_{k=1}^{n} (T)_{j,k}(\mathbf{v})_k > 0$$

for each integer j between 1 and m, because $(T)_{j,k}(\mathbf{v})_k \geq 0$ for $k = 1, 2, \ldots, n$ and $(T)_{j,k}(\mathbf{v})_k > 0$ for at least one integer k between 1 and n. Therefore $T\mathbf{v} >> \mathbf{0}$.

Conversely suppose that T is an $m \times n$ matrix with with real coefficients which has the property that if and only if $T\mathbf{v} >> \mathbf{0}$ for all non-zero n-dimensional vectors $\mathbf{v}$ with non-negative real coefficients. Let $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ be the standard basis of $\mathbb{R}^n$ with

$$\mathbf{e}_1 = (1, 0, \ldots, 0), \quad \mathbf{e}_2 = (0, 1, \ldots, 0), \ldots, \quad \mathbf{0}_n = (0, 0, \ldots, 1).$$

Then $T\mathbf{e}_k >> \mathbf{0}$ for $k = 1, 2, \ldots, n$, and therefore $(T)_{j,k} = (T\mathbf{e}_k)_j > 0$ for $j = 1, 2, \ldots, m$ and $k = 1, 2, \ldots, n$. The result follows. $\blacksquare$

Proposition 6.3

Let T be a non-negative $n \times n$ (square) matrix. Then there exists a well-defined non-negative real number μ (referred to as the Perron root of T) that may be characterized as the greatest real number ρ for which there exists a non-zero vector $\mathbf{v}$ with real coefficients satisfying the conditions $\mathbf{v} \geq \mathbf{0}$ and $T\mathbf{v} \geq \rho\mathbf{v}$.

Proof

Let

$$\Delta = \{\mathbf{v} \in \mathbb{R}^n : \mathbf{v} \geq \mathbf{0},\ \sum_{j=1}^{n} (\mathbf{v})_j = 1\},$$

and, for each non-negative real number ρ, let E_ρ be the subset of Δ defined so that

$$E_\rho = \{\mathbf{v} \in \mathbb{R}^n : \mathbf{v} \geq \mathbf{0},\ \sum_{j=1}^{n} (\mathbf{v})_j = 1 \text{ and } T\mathbf{v} \geq \rho\mathbf{v}\}.$$

Clearly $E_0 = \Delta$. Also if ρ exceeds the largest coefficient of the matrix T then clearly E_ρ is the empty set. Let

$$I = \{\rho \in \mathbb{R} : \rho \geq 0 \quad \text{and} \quad E_\rho \neq \emptyset\}.$$

Then I is a non-empty set of real numbers which is bounded above. It follows from the Least Upper Bound Principle that the set I has a least upper bound $\sup I$. Let $\mu = \sup I$.

Let ρ be a real number satisfying $0 \leq \rho < \mu$. Then there exists $\rho' \in I$ satisfying $\rho < \rho' \leq \mu$. The set $E_{\rho'}$ must then be non-empty, and moreover $E_{\rho'} \subset E_\rho$. It follows that $E_\rho \neq \emptyset$, and thus $\rho \in I$. It follows that

$$\{\rho \in \mathbb{R} : 0 \leq \rho < \mu\} \subset I,$$

and thus the subset I of $\mathbb{R}$ is an interval. We next prove that $\mu \in I$.

Now the characterization of the non-negative real number μ as the least upper bound of the interval I ensures the existence of an infinite sequence $\rho_1, \rho_2, \rho_3, \ldots$ of real numbers belonging to I for which $\lim\limits_{s \to +\infty} \rho_s = \mu$. Then $E_{\rho_s} \neq \emptyset$ for all positive integers s, and therefore there exists an infinite sequence $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots$ of vectors belonging to the simplex Δ such that $\mathbf{v}_s \in E_{\rho_s}$ for all positive integers s. Then $T\mathbf{v}_s \geq \rho_s \mathbf{v}_s$ for all positive integers s.

Now the sequence $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots$ of vectors belonging to the simplex Δ is a bounded sequence of vectors, because Δ is a bounded set. The multidimensional Bolzano-Weierstrass Theorem (Theorem 1.4) now ensures the existence of a subsequence $\mathbf{v}_{s_1}, \mathbf{v}_{s_2}, \mathbf{v}_{s_3}, \ldots$ of the sequence $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots$ which converges to some vector $\mathbf{u}$. Moreover $\mathbf{u} \in \Delta$, because Δ is a closed set. Now

$$T\mathbf{u} = \lim_{r \to +\infty} T\mathbf{v}_{s_r}.$$

Also

$$T\mathbf{v}_{s_r} - \rho_{s_r}\mathbf{v}_{s_r} \geq \mathbf{0}$$

for all positive integers r. Taking limits as $r \to +\infty$, we find that

$$T\mathbf{u} - \mu\mathbf{u} \geq 0,$$

and thus $T\mathbf{u} \geq \mu\mathbf{u}$.

The vector $\mathbf{u}$ is then a non-zero vector with non-negative coefficients, and $T\mathbf{u} \geq \rho\mathbf{u}$ for all real numbers ρ satisfying $0 \leq \rho \leq \mu$.

Now every non-zero n-dimensional vector with non-negative real coefficients is a scalar multiple of some vector belonging to the simplex Δ. We conclude therefore that if ρ is a non-negative real number, if $\mathbf{v}$ is a non-zero vector with non-negative coefficients, and if $T\mathbf{v} \geq \rho\mathbf{v}$ then $\rho \leq \mu$. The result follows. ∎

Definition

Let T be a non-negative square matrix. The *Perron root* (or *Perron-Frobenius eigenvalue*) of T is the unique non-negative real number μ of T that can be characterized as the greatest real number for which there exists a non-zero vector $\mathbf{v}$ with real coefficients satisfying the conditions $\mathbf{v} \geq \mathbf{0}$ and $T\mathbf{v} \geq \mu\mathbf{v}$.

Remark

Proposition 6.3 ensures that every non-negative square matrix has a well-defined Perron root. The alternative name *Perron-Frobenius eigenvalue* for the Perron root seems to imply that the Perron root of a non-negative square matrix is an eigenvalue of that matrix. This result is indeed true. It will be proved for positive square matrices in Proposition 6.4, and the result will be extended to non-negative square matrices in Proposition 6.5. The eigenvalues of a square matrix over the field of complex numbers are the roots of the characteristic polynomial of that matrix.

Proposition 6.4

Let T be a positive square matrix, and let μ be the Perron root of T. Then $\mu > 0$, and there exists $\mathbf{b} \in \mathbb{R}^n$ satisfying the conditions $\mathbf{b} >> \mathbf{0}$ and $T\mathbf{b} = \mu\mathbf{b}$. Moreover, given any $\mathbf{u} \in \mathbb{R}^n$ satisfying $T\mathbf{u} \geq \mu\mathbf{u}$, there exists a real number t for which $\mathbf{u} = t\mathbf{b}$, and thus $T\mathbf{u} = \mu\mathbf{u}$.

Proof

The definition of the Perron root μ of T ensures that there exists a non-zero vector $\mathbf{b}$ with the properties that $\mathbf{b} \geq \mathbf{0}$ and $T\mathbf{b} \geq \mu\mathbf{b}$. Suppose it were the case that $T\mathbf{b} \neq \mu\mathbf{b}$. Let $\mathbf{v} = T\mathbf{b}$. Then

$$T\mathbf{v} - \mu\mathbf{v} = T(T\mathbf{b} - \mu\mathbf{b}) >> \mathbf{0},$$

because $T\mathbf{b} - \mu\mathbf{b} \geq \mathbf{0}$, $T\mathbf{b} - \mu\mathbf{b} \neq \mathbf{0}$ and $T >> 0$ (see Lemma 6.2). But then there would exist real numbers ρ satisfying $\rho > \mu$ that were sufficiently close to μ to ensure that $T\mathbf{v} - \rho\mathbf{v} >> \mathbf{0}$ and thus $T\mathbf{v} \geq \rho\mathbf{v}$. This would contradict the condition on the statement of the proposition that characterizes the value of μ. We conclude therefore that $T\mathbf{b} = \mu\mathbf{b}$.

Moreover $T\mathbf{b} >> \mathbf{0}$, because $T >> 0$ and $\mathbf{b} \geq 0$. It follows that $\mu > 0$ and $\mathbf{b} >> \mathbf{0}$.

Next let $\mathbf{u}$ be an n-dimensional vector with real coefficients for which $T\mathbf{u} \geq \mu\mathbf{u}$. If s is positive and sufficiently large then then $s\mathbf{b} - \mathbf{u} >> 0$. On the other hand if s is negative and $|s|$ is sufficiently large then then $s\mathbf{b} - \mathbf{u} << \mathbf{0}$. It follows from this that there exists a well-defined real number t defined so that

$$t = \inf\{s \in \mathbb{R} : s\mathbf{b} - \mathbf{u} \geq \mathbf{0}\}.$$

Then $t\mathbf{b} - \mathbf{u} \geq 0$, and moreover there exists some integer j between 1 and n for which $t(\mathbf{b})_j - (\mathbf{u})_j = 0$. Now

$$T(t\mathbf{b} - \mathbf{u}) = \mu t\mathbf{b} - T\mathbf{u} \leq \mu(t\mathbf{b} - \mathbf{u}),$$

and therefore $(T(t\mathbf{b} - \mathbf{u}))_j \leq 0$. If it were the case that $t\mathbf{b} - \mathbf{u} \neq 0$ then the inequalities $t\mathbf{b} - \mathbf{u} \geq 0$ and $T >> 0$ would ensure that $T(t\mathbf{b} - \mathbf{b}) >> \mathbf{0}$ (Lemma 6.2), from which it would follow that $(T(t\mathbf{b} - \mathbf{b}))_j > 0$. Because this latter inequality does not hold, it must be the case that $t\mathbf{b} - \mathbf{u} = 0$, and thus $\mathbf{u} = t\mathbf{b}$. The result follows. ∎

Proposition 6.5

Let T be a non-negative square matrix, and let μ be the Perron root of T. Then μ is an eigenvalue of T, and there exists a non-negative eigenvector $\mathbf{b}$ associated with the eigenvalue μ.

Proof

Let T be an $n \times n$ matrix. Then there exists an infinite sequence $T_1, T_2, T_3, \ldots$ of positive $n \times n$ matrices such that $T_r >> T$ for all positive integers r and $T_r \to T$ as $r \to +\infty$. Let μ_r be the Perron root of T_r and let $\mathbf{b}_r$ be the associated positive eigenvector, normalized to satisfy the condition $\sum_{j=1}^{n} (\mathbf{b}_r)_j = 1$.

The multidimensional Bolzano-Weierstrass Theorem (Theorem 1.4) ensures the existence of an infinite subsequence $T_{r_1}, T_{r_2}, T_{r_3}, \ldots$, a real number μ' and a vector $\mathbf{b} \in \mathbb{R}^n$ such that $\mu_{r_s} \to \mu$ and $\mathbf{b}_{r_s} \to \mathbf{b}$. Replacing the original sequence T_1, T_2, T_3 by a subsequence, if necessary, we may assume, without loss of generality, that $\mu_r \to \mu'$ and $\mathbf{b}_r \to \mathbf{b}$ as $r \to +\infty$. Then $\mu' \geq 0$, $(\mathbf{b})_j \geq 0$ for $j = 1, 2, \ldots, n$ and $\sum_{j=1}^{n} (\mathbf{b})_j = 1$. Then

$$T\mathbf{b} - \mu'\mathbf{b} = \lim_{r \to +\infty} (T_r\mathbf{b}_r - \mu_r\mathbf{b}_r) = \mathbf{0}.$$

Thus μ' is an eigenvalue of T, and $\mathbf{b}$ is a non-zero non-negative eigenvector of T associated to the eigenvalue μ'.

It remains to show that $\mu' = \mu$. Let ρ be a non-negative real number. Suppose that there exists a non-zero vector $\mathbf{v} \in \mathbb{R}^n$ such that $\mathbf{v} \geq \mathbf{0}$ and $T\mathbf{v} \geq \rho\mathbf{v}$. Then, for each integer r, $T_r\mathbf{v} \geq \rho\mathbf{v}$, because $T_r >> T$, and therefore $\rho \leq \mu_r$. It follows that $\rho \leq \mu'$, because $\mu' = \lim\limits_{r \to +\infty} \mu_r$. Also $T\mathbf{b} = \mu'\mathbf{b}$. It follows that μ' is the largest real number for which there exists a non-zero vector $\mathbf{v} \in \mathbb{R}^n$ such that $\mathbf{v} \geq \mathbf{0}$ and $T\mathbf{v} \geq \rho\mathbf{v}$. Thus $\mu' = \mu$. The result follows. $\blacksquare$

Lemma 6.6

Let T be a non-negative $n \times n$ (square) matrix, let λ be a complex number, let $\mathbf{u}$ be an non-zero n-dimensional vector with complex coefficients, and let $\mathbf{v}$ be the n-dimensional vector with non-negative real coefficients defined such that $(\mathbf{v})_j = |(\mathbf{u})_j|$ for $j = 1, 2, \ldots, n$. Suppose that $\mathbf{u}$ is an eigenvector of T with eigenvalue λ, so that $T\mathbf{u} = \lambda\mathbf{u}$. Then $T\mathbf{v} \geq |\lambda|\mathbf{v}$. Moreover if $T >> 0$, and if $T\mathbf{v} = |\lambda|\mathbf{v}$, then λ is a positive real number, and there exists some complex number ω satisfying $|\omega| = 1$ for which $\mathbf{u} = \omega\mathbf{v}$.

Proof

There exist real numbers $\theta_1, \theta_2, \ldots, \theta_n$ and φ such that $u_j = e^{i\theta_j} v_j$ for $j = 1, 2, \ldots, n$ and $\lambda = e^{i\varphi}|\lambda|$, where $i = \sqrt{-1}$. (Here $e^{i\alpha} = \cos\alpha + i\sin\alpha$ for all real numbers α.) The identity $T\mathbf{u} = \lambda\mathbf{u}$ ensures that

$$|\lambda| v_j = e^{-i\varphi - i\theta_j} \lambda u_j = e^{-i\varphi - i\theta_j} \sum_{k=1}^{n} T_{j,k} u_k = \sum_{k=1}^{n} e^{-i\varphi + i\theta_k - i\theta_j} T_{j,k} v_k.$$

Taking real parts, we see that

$$|\lambda| v_j = \sum_{k=1}^{n} \cos(-\varphi + \theta_k - \theta_j) T_{j,k} v_k \leq \sum_{k=1}^{n} T_{j,k} v_k.$$

It follows that $T\mathbf{v} \geq |\lambda|\mathbf{v}$. Moreover if $T\mathbf{v} = |\lambda|\mathbf{v}$ then $\cos(-\varphi + \theta_k - \theta_j) = 1$ for all integers j and k between 1 and n for which $v_k > 0$ and $T_{j,k} > 0$.

Now suppose that $T >> 0$ and $T\mathbf{v} = |\lambda|\mathbf{v}$. Then $\mathbf{v} \neq 0$, because $\mathbf{u} \neq 0$. Also $\mathbf{v} \geq 0$. Therefore $T\mathbf{v} >> \mathbf{0}$ (Lemma 6.2). It follows that $\lambda \neq 0$ and $\mathbf{v} >> \mathbf{0}$. Then $T_{j,k} > 0$ and $v_k > 0$ for all integers j and k between 1 and n, and therefore $\cos(-\varphi + \theta_k - \theta_j) = 1$ for all integers j and k. Applying this result with $j = k$, we find that $\cos(-\varphi) = 1$, and therefore φ is an integer multiple of 2π. It then follows that $\theta_j - \theta_k$ is an integer multiple of 2π for all j and k. But these real numbers φ, θ_j and θ_k are only determined up to addition of an integer multiple of 2π. Let $\omega = e^{i\theta_1}$. Then $e^{i\varphi} = 1$ and and $e^{i\theta_j} = \omega$ for $j = 1, 2, \ldots, n$. It follows that λ is real and positive, and $\mathbf{u} = \omega\mathbf{v}$, where ω is a complex number satisfying $|\omega| = 1$, as required. ∎

Proposition 6.7

Let T be a non-negative square matrix, and let μ be the Perron root of T. Then every eigenvalue λ of T satisfies the inequality $|\lambda| \leq \mu$.

Proof

Let λ be an eigenvalue of T, and let $\mathbf{u}$ be an eigenvector of T with eigenvalue λ. The number λ and the coefficients of the vector $\mathbf{u}$ may be real or complex. Let $\mathbf{v} \in \mathbb{R}^n$ be defined such that $(\mathbf{v})_j = |(\mathbf{u})_j)|$ for $j = 1, 2, \ldots, n$. Now $T\mathbf{u} = \lambda\mathbf{u}$. It follows from Lemma 6.6 that $T\mathbf{v} \geq |\lambda|\mathbf{v}$. The definition of the Perron root μ then ensures that $|\lambda| \leq \mu$, as required. ∎