MA3486—Fixed Point Theorems and Economic Equilibria
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5.3. The Kakutani Fixed Point Theorem

Theorem 5.4 (Kakutani's Fixed Point Theorem)

Let X be a non-empty, compact and convex subset of n-dimensional Euclidean space $\mathbb{R}^n$, and let $\Phi\colon X \rightrightarrows X$ be a correspondence mapping X into itself. Suppose that the graph of the correspondence Φ is closed and that $\Phi(\mathbf{x})$ is non-empty and convex for all $\mathbf{x} \in X$. Then there exists a point $\mathbf{x}^*$ of X that satisfies $\mathbf{x}^* \in \Phi(\mathbf{x}^*)$.

Proof

There exists a continuous map $r\colon \mathbb{R}^n \to X$ from $\mathbb{R}^n$ to X with the property that $r(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in X$. (see Proposition 3.8). Let Δ be an n-dimensional simplex chosen such that $X \subset \Delta$, and let $\Psi(\mathbf{x}) = \Phi(r(\mathbf{x}))$ for all $\mathbf{x} \in \Delta$. If $\mathbf{x}^* \in \Delta$ satisfies $\mathbf{x}^* \in \Psi(\mathbf{x}^*)$ then $\mathbf{x}^* \in X$ and $r(\mathbf{x}^*) = \mathbf{x}^*$, and therefore $\mathbf{x} \in \Phi(\mathbf{x}^*)$. It follows that the result in the general case follows from that in the special case in which the closed bounded convex subset X of $\mathbb{R}^n$ is an n-dimensional simplex.

Thus let Δ be an n-dimensional simplex contained in $\mathbb{R}^n$, and let $\Phi\colon \Delta \rightrightarrows \Delta$ be a correspondence with closed graph, where $\Phi(\mathbf{x})$ is a non-empty closed convex subset of Δ for all $\mathbf{x} \in X$. We must prove that there exists some point $\mathbf{x}^*$ of Δ with the property that $\mathbf{x}^* \in \Phi(\mathbf{x}^*)$.

Let K be the simplicial complex consisting of the n-simplex Δ together with all its faces, and let $K^{(j)}$ be the jth barycentric subdivision of K for all positive integers j. Then $|K^{(j)}| = \Delta$ for all positive integers j. Now $\Phi(\mathbf{v})$ is non-empty for all vertices $\mathbf{v}$ of $K^{(j)}$. Now any function mapping the vertices of a simplicial complex into a Euclidean space extends uniquely to a piecewise linear map defined over the polyhedron of that simplicial complex (Proposition 4.8). Therefore there exists a sequence $f_1, f_2, f_3, \ldots$ of continuous functions mapping the simplex Δ into itself such that, for each positive integer j, the continuous map $f_j \colon \Delta \to \Delta$ is piecewise linear on the simplices of $K^{(j)}$ and satisfies $f_j(\mathbf{v}) \in \Phi(\mathbf{v})$ for all vertices $\mathbf{v}$ of $K^{(j)}$.

Now it follows from the Brouwer Fixed Point Theorem (Theorem 5.3) that, for each positive integer j, there exists $\mathbf{z}_j \in \Delta$ for which $f_j(\mathbf{z}_j) = \mathbf{z}_j$. For each positive integer j, there exist vertices

$$\mathbf{v}_{0,j}, \mathbf{v}_{1,j}, \ldots, \mathbf{v}_{n,j}$$

of $K^{(j)}$ spanning a simplex of K and non-negative real numbers $t_{0,j}, t_{1,j}, \ldots, t_{n,j}$ satisfying $\sum\limits_{i=1}^{n} t_{i,j} = 1$ such that

$$\mathbf{z}_j = \sum_{i=0}^{n} t_{i,j} \mathbf{v}_{i,j}$$

for all positive integers j. Let $\mathbf{y}_{i,j} = f_j(\mathbf{v}_{i,j})$ for $i = 0, 1, \ldots, n$ and for all positive integers j. Then $\mathbf{y}_{i,j} \in \Phi(\mathbf{v}_{i,j})$ for $i = 0, 1, \ldots, n$ and for all positive integers j.

The function f_j is piecewise linear on the simplices of $K^{(j)}$. It follows that

$$\begin{aligned}\sum_{i=0}^{n} t_{i,j}\mathbf{v}_{i,j} &= \mathbf{z}_j = f_j(\mathbf{z}_j) = f_j\left(\sum_{i=0}^{n} t_{i,j}\mathbf{v}_{i,j}\right)\\ &= \sum_{i=0}^{n} t_{i,j} f_j(\mathbf{v}_{i,j}) = \sum_{i=0}^{n} t_{i,j}\mathbf{y}_{i,j}\end{aligned}$$

for all positive integers j. Also $|\mathbf{v}_{i,j} - \mathbf{v}_{0,j}| \leq \mu(K^{(j)})$ for $i = 0, 1, \ldots, n$ and for all positive integers j, where $\mu(K^{(j)})$ denotes the mesh of the simplicial complex $K^{(j)}$ (i.e., the length of the longest side of that simplicial complex). Moreover $\mu(K^j) \to 0$ as $j \to +\infty$ (see Lemma 4.6). It follows that

$$\lim_{j\to+\infty} |\mathbf{v}_{i,j} - \mathbf{v}_{0,j}| = 0.$$

Now the multidimensional Bolzano-Weierstrass Theorem (Theorem 1.4) ensures the existence of points

$$\mathbf{x}^*, \mathbf{y}_0, \mathbf{y}_1, \ldots, \mathbf{y}_n$$

of the simplex Δ, non-negative real numbers $t_0, t_1, \ldots, t_n$ and an infinite sequence $m_1, m_2, m_3, \ldots$ of positive integers, where

$$m_1 < m_2 < m_3 < \cdots ,$$

such that

$$\begin{aligned} \mathbf{x}^* &= \lim_{j \to +\infty} \mathbf{v}_{0,m_j}, \\ \mathbf{y}_i &= \lim_{j \to +\infty} \mathbf{y}_{i,m_j} \quad (0 \leq i \leq n), \\ t_i &= \lim_{j \to +\infty} t_{i,m_j} \quad (0 \leq i \leq n). \end{aligned}$$

Now

$$|\mathbf{v}_{i,m_j} - \mathbf{x}^*| \leq |\mathbf{v}_{i,m_j} - \mathbf{v}_{0,m_j}| + |\mathbf{v}_{0,m_j} - \mathbf{x}^*|$$

for $i = 0, 1, \ldots, n$ and for all positive integers j. Moreover $\lim_{j \to +\infty} |\mathbf{v}_{i,m_j} - \mathbf{v}_{0,m_j}| = 0$ and $\lim_{j \to +\infty} |\mathbf{v}_{0,m_j} - \mathbf{x}^*| = 0$. It follows that $\lim_{j \to +\infty} \mathbf{v}_{i,m_j} = \mathbf{x}^*$ for $i = 0, 1, \ldots, n$. Also

$$\sum_{i=0}^{n} t_i = \lim_{j \to +\infty} \left(\sum_{i=0}^{n} t_{i,m_j} \right) = 1.$$

It follows that

$$\begin{aligned} \lim_{j \to +\infty} \left(\sum_{i=0}^{n} t_{i,m_j} \mathbf{v}_{i,m_j} \right) &= \sum_{i=0}^{n} \left(\lim_{j \to +\infty} t_{i,m_j} \right) \left(\lim_{j \to +\infty} \mathbf{v}_{i,m_j} \right) \\ &= \sum_{i=0}^{n} t_i \mathbf{x}^* = \mathbf{x}^*. \end{aligned}$$

But we have also shown that $\sum_{i=0}^{n} t_{i,j}\mathbf{y}_{i,j} = \sum_{i=0}^{n} t_{i,j}\mathbf{v}_{i,j}$ for all positive integers j. It follows that

$$\sum_{i=0}^{n} t_i \mathbf{y}_i = \lim_{j\to+\infty}\left(\sum_{i=0}^{n} t_{i,m_j}\mathbf{y}_{i,m_j}\right) = \lim_{j\to+\infty}\left(\sum_{i=0}^{n} t_{i,m_j}\mathbf{v}_{i,m_j}\right) = \mathbf{x}^*.$$

Next we show that $\mathbf{y}_i \in \Phi(\mathbf{x}^*)$ for $i = 0, 1, \ldots, n$. Now

$$(\mathbf{v}_{i,m_j}, \mathbf{y}_{i,m_j}) \in \mathrm{Graph}(\Phi)$$

for all positive integers j, and the graph $\mathrm{Graph}(\Phi)$ of the correspondence Φ is closed. It follows that

$$(\mathbf{x}^*, \mathbf{y}_i) = \lim_{j \to +\infty} (\mathbf{v}_{i,m_j}, \mathbf{y}_{i,m_j}) \in \mathrm{Graph}(\Phi)$$

and thus $\mathbf{y}_i \in \Phi(\mathbf{x}^*)$ for $i = 0, 1, \ldots, m$ (see Proposition 2.6).

It follows from the convexity of $\Phi(\mathbf{x}^*)$ that

$$\sum_{i=0}^{n} t_i \mathbf{y}_* \in \Phi(\mathbf{x}^*).$$

(see Lemma 3.5). But $\sum_{i=0}^{n} t_i \mathbf{y}_* = \mathbf{x}^*$. It follows that $\mathbf{x}^* \in \Phi(\mathbf{x}^*)$, as required. ∎