MA3486—Fixed Point Theorems and Economic Equilibria
School of Mathematics, Trinity College
Hilary Term 2018
Lecture 15 (February 16, 2018)

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5. Fixed Point Theorems

5.1. Sperner's Lemma

Definition

Let K be a simplicial complex which is a subdivision of some n-dimensional simplex Δ. We define a *Sperner labelling* of the vertices of K to be a function, labelling each vertex of K with an integer between 0 and n, with the following properties:—

- for each $j \in \{0, 1, \ldots, n\}$, there is exactly one vertex of Δ labelled by j,
- if a vertex $\mathbf{v}$ of K belongs to some face of Δ, then some vertex of that face has the same label as $\mathbf{v}$.

Lemma 5.1 (Sperner's Lemma)

Let K be a simplicial complex which is a subdivision of an n-simplex Δ. Then, for any Sperner labelling of the vertices of K, the number of n-simplices of K whose vertices are labelled by $0, 1, \ldots, n$ is odd.

Proof

Given integers $i_0, i_1, \ldots, i_q$ between 0 and n, let $N(i_0, i_1, \ldots, i_q)$ denote the number of q-simplices of K whose vertices are labelled by $i_0, i_1, \ldots, i_q$ (where an integer occurring k times in the list labels exactly k vertices of the simplex). We must show that $N(0, 1, \ldots, n)$ is odd.

We prove the result by induction on the dimension n of the simplex Δ; it is clearly true when $n = 0$. Suppose that the result holds in dimensions less than n. For each simplex σ of K of dimension n, let $p(\sigma)$ denote the number of $(n-1)$-faces of σ labelled by $0, 1, \ldots, n-1$. If σ is labelled by $0, 1, \ldots, n$ then $p(\sigma) = 1$; if σ is labelled by $0, 1, \ldots, n-1, j$, where $j < n$, then $p(\sigma) = 2$; in all other cases $p(\sigma) = 0$. Therefore

$$\sum_{\substack{\sigma \in K \\ \dim \sigma = n}} p(\sigma) = N(0, 1, \ldots, n) + 2 \sum_{j=0}^{n-1} N(0, 1, \ldots, n-1, j).$$

Now the definition of Sperner labellings ensures that the only $(n-1)$-face of Δ containing simplices of K labelled by $0, 1, \ldots, n-1$ is that with vertices labelled by $0, 1, \ldots, n-1$.

Thus if M is the number of $(n-1)$-simplices of K labelled by $0, 1, \ldots, n-1$ that are contained in this face, then $N(0, 1, \ldots, n-1) - M$ is the number of $(n-1)$-simplices labelled by $0, 1, \ldots, n-1$ that intersect the interior of Δ. It follows that

$$\sum_{\substack{\sigma \in K \\ \dim \sigma = n}} p(\sigma) = M + 2\bigl(N(0, 1, \ldots, n-1) - M\bigr),$$

since any $(n-1)$-simplex of K that is contained in a proper face of Δ must be a face of exactly one n-simplex of K, and any $(n-1)$-simplex that intersects the interior of Δ must be a face of exactly two n-simplices of K. On combining these equalities, we see that $N(0, 1, \ldots, n) - M$ is an even integer. But the induction hypothesis ensures that Sperner's Lemma holds in dimension $n-1$, and thus M is odd. It follows that $N(0, 1, \ldots, n)$ is odd, as required. ∎

5.2. Proof of Brouwer's Fixed Point Theorem

Proposition 5.2

Let Δ be an n-simplex with boundary $\partial\Delta$. Then there does not exist any continuous map $r\colon \Delta \to \partial\Delta$ with the property that $r(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in \partial\Delta$.

Proof

Suppose that such a map $r\colon \Delta \to \partial\Delta$ were to exist. It would then follow from the Simplicial Approximation Theorem (Theorem 4.14) that there would exist a simplicial approximation $s\colon K \to L$ to the map r, where L is the simplicial complex consisting of all of the proper faces of Δ, and K is the jth barycentric subdivision, for some sufficiently large j, of the simplicial complex consisting of the simplex Δ together with all of its faces.

If $\mathbf{v}$ is a vertex of K belonging to some proper face Σ of Δ then $r(\mathbf{v}) = \mathbf{v}$, and hence $s(\mathbf{v})$ must be a vertex of Σ, since $s\colon K \to L$ is a simplicial approximation to $r\colon \Delta \to \partial\Delta$. In particular $s(\mathbf{v}) = \mathbf{v}$ for all vertices $\mathbf{v}$ of Δ. Thus if $\mathbf{v} \mapsto m(\mathbf{v})$ is a labelling of the vertices of Δ by the integers $0, 1, \ldots, n$, then $\mathbf{v} \mapsto m(s(\mathbf{v}))$ is a Sperner labelling of the vertices of K. Thus Sperner's Lemma (Lemma 5.1) guarantees the existence of at least one n-simplex σ of K labelled by $0, 1, \ldots, n$. But then $s(\sigma) = \Delta$, which is impossible, since Δ is not a simplex of L. We conclude therefore that there cannot exist any continuous map $r\colon \Delta \to \partial\Delta$ satisfying $r(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in \partial\Delta$. $\blacksquare$

Theorem 5.3 (Brouwer Fixed Point Theorem)

(Brouwer Fixed Point Theorem) Let X be a subset of a Euclidean space that is homeomorphic to the closed n-dimensional ball E^n, where

$$E^n = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x}| \leq 1\}.$$

Then any continuous function $f : X \to X$ mapping the set X into itself has at least one fixed point $\mathbf{x}^*$ for which $f(\mathbf{x}^*) = \mathbf{x}^*$.

Proof

The closed n-dimensional ball E^n is itself homeomorphic to an n-dimensional simplex Δ. Therefore there exists a homeomorphism $h\colon X \to \Delta$ mapping the set X onto the simplex Δ. Then the continuous map $f\colon X \to X$ determines a continuous map $g\colon \Delta \to \Delta$, where $g(h(\mathbf{x})) = h(f(\mathbf{x}))$ for all $\mathbf{x} \in X$. Suppose that it were the case that $f(\mathbf{x}) \neq \mathbf{x}$ for all $\mathbf{x} \in X$. Then $g(\mathbf{z}) \neq \mathbf{z}$ for all $\mathbf{z} \in \Delta$. There would then exist a well-defined continuous map $r\colon \Delta \to \partial\Delta$ mapping each point $\mathbf{z}$ of Δ to the unique point $r(\mathbf{z})$ of the boundary $\partial\Delta$ of Δ at which the half line starting at $g(\mathbf{z})$ and passing through $\mathbf{z}$ intersects $\partial\Delta$. Then $r\colon \Delta \to \partial\Delta$ would be continuous, and $r(\mathbf{z}) = \mathbf{z}$ for all $\mathbf{z} \in \partial\Delta$. However Proposition 5.2 guarantees that there does not exist any continuous map $r\colon \Delta \to \partial\Delta$ with these properties. Therefore the map f must have at least one fixed point, as required. ∎