MA3486—Fixed Point Theorems and Economic Equilibria
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David R. Wilkins

4. Simplicial Complexes

4.1. Simplical Complexes in Euclidean Spaces

Definition

A finite collection K of simplices in $\mathbb{R}^k$ is said to be a *simplicial complex* if the following two conditions are satisfied:—

- if σ is a simplex belonging to K then every face of σ also belongs to K,
- if σ_1 and σ_2 are simplices belonging to K then either $\sigma_1 \cap \sigma_2 = \emptyset$ or else $\sigma_1 \cap \sigma_2$ is a common face of both σ_1 and σ_2.

Definition

The *dimension* of a simplicial complex K is the greatest non-negative integer n with the property that K contains an n-simplex.

Definition

The *polyhedron* of a simplicial complex K is the union of all the simplices of K.

The polyhedron $|K|$ of a simplicial complex K is a subset of a Euclidean space that is both closed and bounded. It is therefore a compact subset of that Euclidean space.

Example

Let K_σ consist of some n-simplex σ together with all of its faces. Then K_σ is a simplicial complex of dimension n, and $|K_\sigma| = \sigma$.

Lemma 4.1

Let K be a simplicial complex, and let X be a subset of some Euclidean space. A function $f\colon |K| \to X$ is continuous on the polyhedron $|K|$ of K if and only if the restriction of f to each simplex of K is continuous on that simplex.

Proof

Each simplex of the simplicial complex K is a closed subset of the polyhedron $|K|$ of the simplicial complex K. The numbers of simplices belonging to the simplicial complex is finite. The result therefore follows from a straightforward application of Lemma 1.19. ∎

We shall denote by $\mathrm{Vert}\, K$ the set of vertices of a simplicial complex K (i.e., the set consisting of all vertices of all simplices belonging to K). A collection of vertices of K is said to *span* a simplex of K if these vertices are the vertices of some simplex belonging to K.

Definition

Let K be a simplicial complex in $\mathbb{R}^k$. A *subcomplex* of K is a collection L of simplices belonging to K with the following property:—

- if σ is a simplex belonging to L then every face of σ also belongs to L.

Note that every subcomplex of a simplicial complex K is itself a simplicial complex.

Proposition 4.2

Let K be a finite collection of simplices in some Euclidean space $\mathbb{R}^k$, and let $|K|$ be the union of all the simplices in K. Then K is a simplicial complex (with polyhedron $|K|$) if and only if the following two conditions are satisfied:—

- *K contains the faces of its simplices,*
- *every point of $|K|$ belongs to the interior of a unique simplex of K.*

Proof

Suppose that K is a simplicial complex. Then K contains the faces of its simplices. We must show that every point of $|K|$ belongs to the interior of a unique simplex of K. Let $\mathbf{x} \in |K|$. Then $\mathbf{x} \in \rho$ for some simplex ρ of K. It follows from Lemma 3.3 that there exists a unique face σ of ρ such that the point $\mathbf{x}$ belongs to the interior of σ. But then $\sigma \in K$, because $\rho \in K$ and K contains the faces of all its simplices. Thus $\mathbf{x}$ belongs to the interior of at least one simplex of K.

Suppose that **x** were to belong to the interior of two distinct simplices σ and τ of K. Then **x** would belong to some common face $\sigma \cap \tau$ of σ and τ (since K is a simplicial complex). But this common face would be a proper face of one or other of the simplices σ and τ (since $\sigma \neq \tau$), contradicting the fact that **x** belongs to the interior of both σ and τ. We conclude that the simplex σ of K containing **x** in its interior is uniquely determined.

Conversely, we must show that if K is some finite collection of simplices in some Euclidean space, if K contains the faces of all its simplices, and if every point of the union $|K|$ of those simplices belongs the the interior of a unique simplex in the collection, then that collection is a simplicial complex. To achieve this, we must prove that if σ and τ are simplices belonging to the collection K, and if $\sigma \cap \tau \neq \emptyset$, then $\sigma \cap \tau$ is a common face of σ and τ.

Let $\mathbf{x} \in \sigma \cap \tau$. Then $\mathbf{x}$ belongs to the interior of a unique simplex ω belonging to the collection K. However any point of σ or τ belongs to the interior of a unique face of that simplex, and all faces of σ and τ belong to K. It follows that ω is a common face of σ and τ, and thus the vertices of ω are vertices of both σ and τ. It follows that the simplices σ and τ have vertices in common.

Let ρ be the simplex whose vertex set is the intersection of the vertex sets of σ and τ. Then ρ is a common face of both σ and τ, and therefore $\rho \in K$. Moreover if $\mathbf{x} \in \sigma \cap \tau$ and if ω is the unique simplex of K whose interior contains the point $\mathbf{x}$, then (as we have already shown), all vertices of ω are vertices of both σ and τ. But then the vertex set of ω is a subset of the vertex set of ρ, and thus ω is a face of ρ. Thus each point $\mathbf{x}$ of $\sigma \cap \tau$ belongs to ρ, and therefore $\sigma \cap \tau \subset \rho$. But ρ is a common face of σ and τ and therefore $\rho \subset \sigma \cap \tau$. It follows that $\sigma \cap \tau = \rho$, and thus $\sigma \cap \tau$ is a common face of σ and τ. This completes the proof that the collection K of simplices satisfying the given conditions is a simplicial complex. ∎

4.2. Barycentric Subdivision of a Simplicial Complex

Let σ be a q-simplex in $\mathbb{R}^k$ with vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$. The *barycentre* of σ is defined to be the point

$$\hat{\sigma} = \frac{1}{q+1}(\mathbf{v}_0 + \mathbf{v}_1 + \cdots + \mathbf{v}_q).$$

Let σ and τ be simplices in some Euclidean space. If σ is a proper face of τ then we denote this fact by writing $\sigma < \tau$.

A simplicial complex K_1 is said to be a *subdivision* of a simplicial complex K if $|K_1| = |K|$ and each simplex of K_1 is contained in a simplex of K.

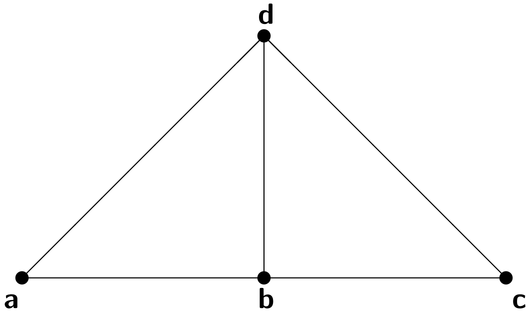
Definition

Let K be a simplicial complex in some Euclidean space $\mathbb{R}^k$. The *first barycentric subdivision* K' of K is defined to be the collection of simplices in $\mathbb{R}^k$ whose vertices are $\hat{\sigma}_0, \hat{\sigma}_1, \ldots, \hat{\sigma}_r$ for some sequence $\sigma_0, \sigma_1, \ldots, \sigma_r$ of simplices of K with $\sigma_0 < \sigma_1 < \cdots < \sigma_r$. Thus the set of vertices of K' is the set of all the barycentres of all the simplices of K.

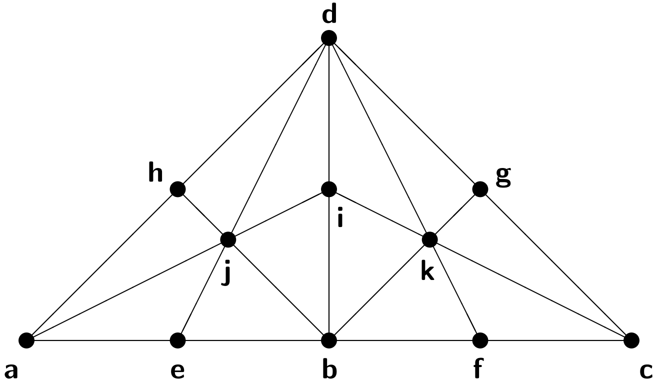
Note that every simplex of K' is contained in a simplex of K. Indeed if $\sigma_0, \sigma_1, \ldots, \sigma_r \in K$ satisfy $\sigma_0 < \sigma_1 < \cdots < \sigma_r$ then the simplex of K' spanned by $\hat{\sigma}_0, \hat{\sigma}_1, \ldots, \hat{\sigma}_r$, is contained in the simplex σ_r of K.

Example

Let K be the simplicial complex consisting of two triangles $\mathbf{a}\,\mathbf{b}\,\mathbf{d}$ and $\mathbf{b}\,\mathbf{c}\,\mathbf{d}$ that intersect along a common edge $\mathbf{b}\,\mathbf{d}$, together with all the edges and vertices of the two triangles, as depicted in the following diagram:



The barycentric subdivision K' of this simplicial complex is then as depicted in the following diagram:



We see that K' consists of 12 triangles, together with all the edges and vertices of those triangles. Of the 11 vertices of K', the vertices **a**, **b**, **c** and **d** are the vertices of the original complex K, the vertices **e**, **f**, **g**, **h** and **i** are the barycentres of the edges **a b**, **b c**, **c d**, **a d** and **b d** respectively, and are located at the midpoints of those edges, and the vertices **j** and **k** are the barycentres of the triangles **a b d** and **b c d** of K. Thus $\mathbf{e} = \frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{b}$, $\mathbf{f} = \frac{1}{2}\mathbf{b} + \frac{1}{2}\mathbf{c}$, etc., and $\mathbf{j} = \frac{1}{3}\mathbf{a} + \frac{1}{3}\mathbf{b} + \frac{1}{3}\mathbf{d}$ and $\mathbf{k} = \frac{1}{3}\mathbf{b} + \frac{1}{3}\mathbf{c} + \frac{1}{3}\mathbf{d}$.

Proposition 4.3

Let K be a simplicial complex in some Euclidean space, and let K' be the first barycentric subdivision of K. Then K' is itself a simplicial complex, and $|K'| = |K|$.

Proof

We prove the result by induction on the number of simplices in K. The result is clear when K consists of a single simplex, since that simplex must then be a point and therefore $K' = K$. We prove the result for a simplicial complex K, assuming that it holds for all complexes with fewer simplices.

It is clear from the definition of the barycentric subdivision K' that any face of a simplex of K' must itself belong to K'. We must verify that any two simplices of K' are disjoint or else intersect in a common face.

Choose a simplex σ of K for which $\dim \sigma = \dim K$, and let $L = K \setminus \{\sigma\}$. Then L is a subcomplex of K, since σ is not a proper face of any simplex of K. Now L has fewer simplices than K. It follows from the induction hypothesis that L' is a simplicial complex and $|L'| = |L|$. Also it follows from the definition of K' that K' consists of the following simplices:—

- the simplices of L',
- the barycentre $\hat{\sigma}$ of σ,
- simplices $\hat{\sigma}\rho$ whose vertex set is obtained by adjoining $\hat{\sigma}$ to the vertex set of some simplex ρ of L', where the vertices of ρ are barycentres of proper faces of σ.

By checking all possible intersections of simplices of the above types, it is easy to verify that any two simplices of K' intersect in a common face. Indeed any two simplices of L' intersect in a common face, since L' is a simplicial complex. If ρ_1 and ρ_2 are simplices of L' whose vertices are barycentres of proper faces of σ, then $\rho_1 \cap \rho_2$ is a common face of ρ_1 and ρ_2 which is of this type, and $\hat{\sigma}\rho_1 \cap \hat{\sigma}\rho_2 = \hat{\sigma}(\rho_1 \cap \rho_2)$. Thus $\hat{\sigma}\rho_1 \cap \hat{\sigma}\rho_2$ is a common face of $\hat{\sigma}\rho_1$ and $\hat{\sigma}\rho_2$. Also any simplex τ of L' is disjoint from the barycentre $\hat{\sigma}$ of σ, and $\hat{\sigma}\rho \cap \tau = \rho \cap \tau$. We conclude that K' is indeed a simplicial complex.

It remains to verify that $|K'| = |K|$. Now $|K'| \subset |K|$, since every simplex of K' is contained in a simplex of K. Let $\mathbf{x}$ be a point of the chosen simplex σ. Then there exists a point $\mathbf{y}$ belonging to a proper face of σ and some $t \in [0, 1]$ such that $\mathbf{x} = (1 - t)\hat{\sigma} + t\,\mathbf{y}$. But then $\mathbf{y} \in |L|$, and $|L| = |L'|$ by the induction hypothesis. It follows that $\mathbf{y} \in \rho$ for some simplex ρ of L' whose vertices are barycentres of proper faces of σ. But then $\mathbf{x} \in \hat{\sigma}\rho$, and therefore $\mathbf{x} \in |K'|$. Thus $|K| \subset |K'|$, and hence $|K'| = |K|$, as required. ∎

We define (by induction on j) the jth barycentric subdivision $K^{(j)}$ of K to be the first barycentric subdivision of $K^{(j-1)}$ for each $j > 1$.