MA3486—Fixed Point Theorems and Economic Equilibria
School of Mathematics, Trinity College
Hilary Term 2018
Lecture 12 (February 9, 2018)

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3.7. Convex Sets and Supporting Hyperplanes

Lemma 3.9

Let m be a positive integer, let F be a non-empty closed set in $\mathbb{R}^m$, and let $\mathbf{b}$ be a vector in $\mathbb{R}^m$. Then there exists an element $\mathbf{g}$ of F such that $|\mathbf{x} - \mathbf{b}| \geq |\mathbf{g} - \mathbf{b}|$ for all $\mathbf{x} \in F$.

Proof

Let R be a positive real number chosen large enough to ensure that the set F_0 is non-empty, where

$$F_0 = F \cap \{\mathbf{x} \in \mathbb{R}^m : |\mathbf{x} - \mathbf{b}| \leq R\}.$$

Then F_0 is a closed bounded subset of $\mathbb{R}^m$. Let $f \colon F_0 \to \mathbb{R}$ be defined such that $f(\mathbf{x}) = |\mathbf{x} - \mathbf{b}|$ for all $\mathbf{x} \in F$. Then $f \colon F_0 \to \mathbb{R}$ is a continuous function on F_0.

Now it is a standard result of real analysis that any continuous real-valued function on a closed bounded subset of a finite-dimensional Euclidean space attains a minimum value at some point of that set. It follows that there exists an element $\mathbf{g}$ of F_0 such that

$$|\mathbf{x} - \mathbf{b}| \geq |\mathbf{g} - \mathbf{b}|$$

for all $\mathbf{x} \in F_0$. If $\mathbf{x} \in F \setminus F_0$ then

$$|\mathbf{x} - \mathbf{b}| \geq R \geq |\mathbf{g} - \mathbf{b}|.$$

It follows that

$$|\mathbf{x} - \mathbf{b}| \geq |\mathbf{g} - \mathbf{b}|$$

for all $\mathbf{x} \in F$, as required. ∎

3.8. A Separating Hyperplane Theorem

Theorem 3.10

Let m be a positive integer, let X be a closed convex set in $\mathbb{R}^m$, and let $\mathbf{b}$ be point of $\mathbb{R}^m$, where $\mathbf{b} \notin X$. Then there exists a linear functional $\varphi\colon \mathbb{R}^m \to \mathbb{R}$ and a real number c such that $\varphi(\mathbf{x}) > c$ for all $\mathbf{x} \in X$ and $\varphi(\mathbf{b}) < c$.

Proof

It follows from Lemma 3.9 that there exists a point $\mathbf{g}$ of X such that $|\mathbf{x} - \mathbf{b}| \geq |\mathbf{g} - \mathbf{b}|$ for all $\mathbf{x} \in X$. Let $\mathbf{x} \in X$. Then $(1 - t)\mathbf{g} + t\mathbf{x} \in X$ for all real numbers t satisfying $0 \leq t \leq 1$, because the set X is convex, and therefore

$$|(1 - t)\mathbf{g} + t\mathbf{x} - \mathbf{b}| \geq |\mathbf{g} - \mathbf{b}|$$

for all real numbers t satisfying $0 \leq t \leq 1$.

Now

$$(1-t)\mathbf{g} + t\mathbf{x} - \mathbf{b} = \mathbf{g} - \mathbf{b} + t(\mathbf{x} - \mathbf{g}).$$

It follows by a straightforward calculation from the definition of the Euclidean norm that

$$\begin{aligned} |\mathbf{g} - \mathbf{b}|^2 &\leq |(1-t)\mathbf{g} + t\mathbf{x} - \mathbf{b}|^2 \\ &= |\mathbf{g} - \mathbf{b}|^2 + 2t(\mathbf{g} - \mathbf{b}) \,.\, (\mathbf{x} - \mathbf{g}) \\ &\qquad + t^2 |\mathbf{x} - \mathbf{g}|^2 \end{aligned}$$

for all real numbers t satisfying $0 \leq t \leq 1$. In particular, this inequality holds for all sufficiently small positive values of t, and therefore

$$(\mathbf{g} - \mathbf{b}) \,.\, (\mathbf{x} - \mathbf{g}) \geq 0$$

for all $\mathbf{x} \in X$.

Let

$$\varphi(\mathbf{x}) = (\mathbf{g} - \mathbf{b}) \,.\, \mathbf{x}$$

for all $\mathbf{x} \in \mathbb{R}^m$. Then $\varphi\colon \mathbb{R}^m \to \mathbb{R}$ is a linear functional on $\mathbb{R}^m$, and $\varphi(\mathbf{x}) \geq \varphi(\mathbf{g})$ for all $\mathbf{x} \in X$. Moreover

$$\varphi(\mathbf{g}) - \varphi(\mathbf{b}) = |\mathbf{g} - \mathbf{b}|^2 > 0,$$

and therefore $\varphi(\mathbf{g}) > \varphi(\mathbf{b})$. It follows that $\varphi(\mathbf{x}) > c$ for all $\mathbf{x} \in X$, where $c = \frac{1}{2}\varphi(\mathbf{b}) + \frac{1}{2}\varphi(\mathbf{g})$, and that $\varphi(\mathbf{b}) < c$. The result follows. ∎

Let X be a subset of n-dimensional Euclidean space $\mathbb{R}^n$. A point $\mathbf{b}$ lies on the *boundary* of X if every open ball of positive radius centred on the point $\mathbf{b}$ intersects both the set X itself and the complement $\mathbb{R}^n \setminus X$ of X in $\mathbb{R}^n$.
If a subset X of $\mathbb{R}^n$ is open in $\mathbb{R}^n$ then every point belonging to the boundary of the set X belongs to the complement of X. If the subset X of $\mathbb{R}^m$ is closed in $\mathbb{R}^m$ then every point belonging to the boundary of the set X belongs to the set X itself.

Theorem 3.11 (Supporting Hyperplane Theorem)

Let m be a positive integer, let X be a closed convex set in $\mathbb{R}^m$, and let $\mathbf{b}$ be point of $\mathbb{R}^m$ that belongs to the boundary of the closed convex set X. Then there exists a linear functional $\varphi\colon \mathbb{R}^m \to \mathbb{R}$ and a real number c such that $\varphi(\mathbf{x}) \geq c$ for all $\mathbf{x} \in X$ and $\varphi(\mathbf{b}) = c$.

Proof

We may assume without loss of generality, that $\mathbf{b} = (0, 0, \ldots, 0)$. We must then prove the existence of a linear functional $\varphi\colon \mathbb{R}^m \to \mathbb{R}$ with the property that $\varphi(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in X$.

Now, because the $\mathbf{b}$ is located on the boundary of the set X, there exists an infinite sequence $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \ldots$ of points of the complement $\mathbb{R}^n \setminus X$ of the set X that converges to $\mathbf{b}$. It follows from basic linear algebra that, given any linear functional $\psi\colon \mathbb{R}^n \to \mathbb{R}$ on $\mathbb{R}^n$, there exists a vector $\mathbf{w}$ in $\mathbb{R}^n$ such that $\psi(\mathbf{x}) = \mathbf{w} \,.\, \mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$. It therefore follows from Theorem 3.10, that there exists an infinite sequence $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots$ of non-zero vectors in $\mathbb{R}^n$ such that $\mathbf{v}_j \,.\, \mathbf{b}_j < 0$ and $\mathbf{v}_j \,.\, \mathbf{x} \geq 0$ for all $\mathbf{x} \in X$. We may assume, without loss of generality, that $|\mathbf{v}_j| = 1$ for all positive integers j.

It follows from the Bolzano-Weierstrass Theorem (Theorem 1.4) that the infinite sequence $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots$ has a convergent subsequence $\mathbf{v}_{k_1}, \mathbf{v}_{k_2}, \mathbf{v}_{k_3}, \ldots$, where

$$k_1 < k_2 < k_3 < \cdots .$$

Let $\mathbf{v} = \lim_{j \to +\infty} \mathbf{v}_{k_j}$. Then $|\mathbf{v}| = 1$. Let $\varphi(\mathbf{x}) = \mathbf{v} \,.\, \mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$. Then

$$\varphi(\mathbf{x}) = \lim_{j \to +\infty} \mathbf{v}_{k_j} \,.\, \mathbf{x} \geq 0$$

for all $\mathbf{x} \in X$. The result follows. ∎