MA3486—Fixed Point Theorems and Economic Equilibria
School of Mathematics, Trinity College
Hilary Term 2018
Lecture 11 (February 9, 2018)

David R. Wilkins

3.5. The Interior of a Simplex

Definition

The *interior* of a simplex σ is defined to be the set consisting of all points of σ that do not belong to any proper face of σ.

Lemma 3.2

Let σ be a q-simplex in some Euclidean space with vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$. Let $\mathbf{x}$ be a point of σ, and let $t_0, t_1, \ldots, t_q$ be the barycentric coordinates of the point $\mathbf{x}$ with respect to $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$, so that $t_j \geq 0$ for $j = 0, 1, \ldots, q$, $\mathbf{x} = \sum_{j=0}^{q} t_j \mathbf{v}_j$, and $\sum_{j=0}^{q} t_j = 1$. Then the point $\mathbf{x}$ belongs to the interior of σ if and only if $t_j > 0$ for $j = 0, 1, \ldots, q$.

Proof

The point $\mathbf{x}$ belongs to the face of σ spanned by vertices

$$\mathbf{v}_{j_0}, \mathbf{v}_{j_1}, \ldots, \mathbf{v}_{j_r},$$

where $0 \leq j_0 < j_1 < \cdots < j_r \leq q$, if and only if $t_j = 0$ for all integers j between 0 and q that do not belong to the set $\{j_0, j_1, \ldots, j_r\}$. Thus the point $\mathbf{x}$ belongs to a proper face of the simplex σ if and only if at least one of the barycentric coordinates t_j of that point is equal to zero. The result follows. ∎

Example

A 0-simplex consists of a single vertex $\mathbf{v}$. The interior of that 0-simplex is the vertex $\mathbf{v}$ itself.

Example

A 1-simplex is a line segment. The interior of a line segment in a Euclidean space $\mathbb{R}^k$ with endpoints $\mathbf{v}$ and $\mathbf{w}$ is

$$\{(1-t)\,\mathbf{v} + t\,\mathbf{w} : 0 < t < 1\}.$$

Thus the interior of the line segment consists of all points of the line segement that are not endpoints of the line segment.

Example

A 2-simplex is a triangle. The interior of a triangle with vertices $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ is the set

$$\{r\,\mathbf{u} + s\,\mathbf{v} + t\,\mathbf{w} : 0 < r, s, t < 1 \text{ and } r + s + t = 1\}.$$

The interior of this triangle consists of all points of the triangle that do not lie on any edge of the triangle.

Remark

Let σ be a q-dimensional simplex in some Euclidean space $\mathbb{R}^k$, where $k \geq q$. If $k > q$ then the interior of the simplex (defined according to the definition given above) will not coincide with the topological interior determined by the usual topology on $\mathbb{R}^k$. Consider for example a triangle embedded in three-dimensional Euclidean space $\mathbb{R}^3$. The interior of the triangle (defined according to the definition given above) consists of all points of the triangle that do not lie on any edge of the triangle. But of course no three-dimensional ball of positive radius centred on any point of that triangle is wholly contained within the triangle. It follows that the topological interior of the triangle is the empty set when that triangle is considered as a subset of three-dimensional space $\mathbb{R}^3$.

Lemma 3.3

Any point of a simplex belongs to the interior of a unique face of that simplex.

Proof

let $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ be the vertices of a simplex σ, and let $\mathbf{x} \in \sigma$. Then $\mathbf{x} = \sum_{j=0}^{q} t_j \mathbf{v}_j$, where $t_0, t_1, \ldots, t_q$ are the barycentric coordinates of the point $\mathbf{x}$. Moreover $0 \leq t_j \leq 1$ for $j = 0, 1, \ldots, q$ and $\sum_{j=0}^{q} t_j = 1$. The unique face of σ containing $\mathbf{x}$ in its interior is then the face spanned by those vertices $\mathbf{v}_j$ for which $t_j > 0$. ∎

3.6. Convex Subsets of Euclidean Spaces

Definition

A subset X of n-dimensional Euclidean space $\mathbb{R}^n$ is said to be *convex* if $(1-t)\mathbf{u} + t\mathbf{v} \in X$ for all points $\mathbf{u}$ and $\mathbf{v}$ of X and for all real numbers t satisfying $0 \leq t \leq 1$.

It follows from the above definition that a subset X of $\mathbb{R}^{\gtrdot}$ is a convex subset of $\mathbb{R}^m$ if and only if, given any two points of X, the line segment joining those two points is wholly contained in X.

Lemma 3.4

An simplex in a Euclidean space is a convex subset of that Euclidean space.

Proof

Let σ be a q-simplex in n-dimensional Euclidean space with vertices $\mathbf{w}_0, \mathbf{w}_1, \ldots, \mathbf{w}_q$, and let $\mathbf{u}$ and $\mathbf{v}$ be points of σ. Then there exist non-negative real numbers $y_0, y_1, \ldots, y_q$ and $z_0, z_1, \ldots, z_q$, where $\sum_{i=0}^{q} y_i = 1$ and $\sum_{i=0}^{q} z_i = 1$, such that

$$\mathbf{u} = \sum_{i=0}^{q} y_i \mathbf{w}_i, \quad \mathbf{v} = \sum_{i=0}^{q} z_i \mathbf{w}_i.$$

Then

$$(1-t)\mathbf{u} + t\mathbf{v} = \sum_{i=0}^{q} ((1-t)y_i + tz_i)\mathbf{w}_i.$$

Moreover $(1-t)y_i + tz_i \geq 0$ for $i = 0, 1, \ldots, q$ and for all real numbers t satisfying $0 \leq t \leq 1$. Also

$$\sum_{i=0}^{q}((1-t)y_i + tz_i) = (1-t)\sum_{i=0}^{q} y_i + t\sum_{i=0}^{q} z_i = 1.$$

It follows that $(1-t)\mathbf{u} + t\mathbf{v} \in \sigma$. Thus σ is a convex subset of $\mathbb{R}^n$. $\blacksquare$

Lemma 3.5

Let X be a convex subset of n-dimensional Euclidean space $\mathbb{R}^n$, and let σ be a simplex contained in $\mathbb{R}^n$. Suppose that the vertices of σ belong to X. Then $\sigma \subset X$.

Proof

We prove the result by induction on the dimension q of the simplex σ. The result is clearly true when $q = 0$, because in that case the simplex σ consists of a single point which is the unique vertex of the simplex.

Thus let σ be a q-dimensional simplex, and suppose that the result is true for all $(q-1)$-dimensional simplices whose vertices belong to the convex set X. Let $\mathbf{w}_0, \mathbf{w}_1, \ldots, \mathbf{w}_q$ be the vertices of σ. Let $\mathbf{x}$ be a point of σ. Then there exist non-negative real numbers $t_0, t_1, \ldots, t_q$ satisfying $\sum_{i=0}^{q} t_i = 1$ such that $\mathbf{x} = \sum_{i=0}^{q} t_i \mathbf{w}_i$. If $t_0 = 1$ then $\mathbf{x} = \mathbf{w}_0$, and therefore $\mathbf{x} \in X$.

It remains to consider the case when $t_0 < 1$. In that case let $s_i = t_i/(1 - t_0)$ for $i = 1, 2, \ldots, q$, and let

$$\mathbf{v} = \sum_{i=1}^{q} s_i \mathbf{w}_i.$$

Now $s_i \geq 0$ for $i = 1, 2, \ldots, q$, and

$$\sum_{i=1}^{q} s_i = \frac{1}{1 - t_0} \sum_{i=1}^{q} t_i = \frac{1}{1 - t_0} \left(\sum_{i=0}^{q} t_i - t_0 \right) = 1,$$

It follows that $\mathbf{v}$ belongs to the proper face of σ that is spanned by the vertices $\mathbf{w}_1, \ldots, \mathbf{w}_q$. The induction hypothesis then ensures that $\mathbf{v} \in X$. But then

$$\mathbf{x} = t_0 \mathbf{w}_0 + (1 - t_0)\mathbf{v},$$

where $\mathbf{w}_0 \in X$ and $\mathbf{v} \in X$ and $0 \leq t_0 \leq 1$. It follows from the convexity of X that $\mathbf{x} \in X$, as required. $\blacksquare$

Let X be a convex set in n-dimensional Euclidean space $\mathbb{R}^{\ltimes}$. A point $\mathbf{x}$ of X is said to belong to the *topological interior* of X if there exists some $\delta > 0$ such that $B(\mathbf{x}, \delta) \subset X$, where

$$B(\mathbf{x}, \delta) = \{\mathbf{x}' \in \mathbb{R}^n : |\mathbf{x}' - \mathbf{x}| < \delta\}.$$

Lemma 3.6

Let X be a convex set in n-dimensional Euclidean space $\mathbb{R}^n$, and let $\mathbf{x} = (1-t)\mathbf{u} + t\mathbf{v}$ where $\mathbf{u}, \mathbf{v} \in X$ and $0 < t < 1$. Suppose that either $\mathbf{u}$ or $\mathbf{v}$ belongs to the topological interior of X. Then $\mathbf{x}$ belongs to the topological interior of X.

Proof

Suppose that $\mathbf{v}$ belongs to the topological interior of X. Then there exists $\delta > 0$ such that $B(\mathbf{v}, \delta) \subset X$, where

$$B(\mathbf{v}, \delta) = \{\mathbf{x}' \in \mathbb{R}^n : |\mathbf{x}' - \mathbf{v}| < \delta\}.$$

We claim that $B(\mathbf{x}, t\delta) \subset X$.

Let $\mathbf{x}' \in B(\mathbf{x}, t\delta)$, and let

$$\mathbf{z} = \frac{1}{t}(\mathbf{x}' - \mathbf{x}).$$

Then $\mathbf{v} + \mathbf{z} \in B(\mathbf{v}, \delta)$ and

$$\mathbf{x}' = (1 - t)\mathbf{u} + t(\mathbf{v} + \mathbf{z}),$$

and therefore $\mathbf{x}' \in X$. This proves the result when $\mathbf{v}$ belongs to the topological interior of X. The result when $\mathbf{u}$ belongs to the topological interior of X then follows on interchanging $\mathbf{u}$ and $\mathbf{v}$ and replacing t by $1 - t$. The result follows. $\blacksquare$

Proposition 3.7

Let X be a closed bounded convex subset of n-dimensional Euclidean space $\mathbb{R}^n$ whose topological interior contains the origin, let S^{n-1} be the unit sphere in $\mathbb{R}^n$, defined such that

$$S^{n-1} = \{\mathbf{u} \in \mathbb{R}^n : |\mathbf{u}| = 1\},$$

and let $\lambda\colon S^{n-1} \to \mathbb{R}$ be the real-valued function on S^{n-1} defined such that

$$\lambda(\mathbf{u}) = \sup\{t \in \mathbf{R} : t\mathbf{u} \in X\}$$

for all $\mathbf{u} \in S^{n-1}$. Then the function $\lambda\colon S^{n-1} \to \mathbb{R}$ is continuous.

Proof

Let $\mathbf{u}_0 \in S^{n-1}$, let $t_0 = \lambda(\mathbf{u}_0)$, and let some positive real number ε be given, where $0 < \varepsilon < t_0$. It follows from Lemma 3.6 that $(t_0 - \varepsilon)\mathbf{u}$ belongs to the topological interior of X. It then follows from the continuity of the function sending $\mathbf{u} \in S^{n-1}$ to $(t_0 - \varepsilon)\mathbf{u}$ that there exists some positive real number δ_1 such that $(t_0 - \varepsilon)\mathbf{u} \in X$ for all $\mathbf{u} \in S^{n-1}$ satisfying $|\mathbf{u} - \mathbf{u}_0| < \delta_1$. Therefore $\lambda(\mathbf{u}) \geq t_0 - \varepsilon$ whenever $|\mathbf{u} - \mathbf{u}_0| < \delta_1$.

Next we note that $(t_0 + \varepsilon)\mathbf{u}_0 \notin X$. Now X is closed in $\mathbb{R}^n$, and therefore the complement $\mathbb{R}^n \setminus X$ of X in $\mathbb{R}^n$ is open. It follows that there exists an open ball of positive radius about the point $(t_0 + \varepsilon)\mathbf{u}_0$ that is wholly contained in the complement of X. It then follows from the continuity of the function sending $\mathbf{u} \in S^{n-1}$ to $(t_0 + \varepsilon)\mathbf{u}$ that there exists some positive real number δ_2 such that $(t_0 + \varepsilon)\mathbf{u} \notin X$ for all $\mathbf{u} \in S^{n-1}$ satisfying $|\mathbf{u} - \mathbf{u}_0| < \delta_2$. It then follows from the convexity of X that $t\mathbf{u} \notin X$ for all positive real numbers t satisfying $t \geq t_0 + \varepsilon$. Therefore $\lambda(\mathbf{u}) \leq t_0 + \varepsilon$ whenever $|\mathbf{u} - \mathbf{u}_0| < \delta_2$. Let δ be the minimum of δ_1 and δ_2. Then $\delta > 0$, and

$$\lambda(\mathbf{u}_0) - \varepsilon \leq \lambda(\mathbf{u}) \leq \lambda(\mathbf{u}_0) + \varepsilon$$

for all $\mathbf{u} \in S^{n-1}$ satisfying $|\mathbf{u} - \mathbf{u}_0| < \delta$. The result follows. $\blacksquare$

Proposition 3.8

Let X be a closed bounded convex subset of n-dimensional Euclidean space $\mathbb{R}^n$. Then there exists a continuous map $r\colon \mathbb{R}^n \to X$ such that $r(\mathbb{R}^n) = X$ and $r(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in X$.

Proof

We first prove the result in the special case in which the convex set X has non-empty topological interior. Without loss of generality, we may assume that the origin of $\mathbb{R}^n$ belongs to the topological interior of X. Let

$$S^{n-1} = \{\mathbf{u} \in \mathbb{R}^n : |\mathbf{u}| = 1\},$$

and let $\lambda\colon S^{n-1} \to \mathbb{R}$ be the real-valued function on S^{n-1} defined such that

$$\lambda(\mathbf{u}) = \sup\{t \in \mathbf{R} : t\mathbf{u} \in X\}$$

for all $\mathbf{u} \in S^{n-1}$. Then the function $\lambda\colon S^{n-1} \to \mathbb{R}$ is continuous (Proposition 3.7).

We may therefore define a function $r\colon \mathbb{R}^n \to X$ such that

$$r(\mathbf{x}) = \begin{cases} \mathbf{x} & \text{if } \mathbf{x} \in X; \\ |\mathbf{x}|^{-1}\lambda(|\mathbf{x}|^{-1}\mathbf{x})\mathbf{x} & \text{if } \mathbf{x} \notin X. \end{cases}$$

Let $\mathbf{x} \in X$ and let $\mathbf{u} = |\mathbf{x}|^{-1}\mathbf{x}$. Then $\mathbf{x} = |\mathbf{x}|\,\mathbf{u}$, $|\mathbf{x}| \leq \lambda(\mathbf{u})$ and $\lambda(\mathbf{u})\mathbf{u} \in X$. It follows from Lemma 3.6 that if $|\mathbf{x}| < \lambda(\mathbf{u})$ then the point $\mathbf{x}$ belongs to the topological interior of $\mathbf{u}$. Thus if the point $\mathbf{x}$ of X belongs to the closure of the complement $\mathbb{R}^n \setminus X$ of X then it does not belong to the topological interior of X, and therefore $|\mathbf{x}| = \lambda(|\mathbf{x}|^{-1}\mathbf{x})$, and therefore

$$\mathbf{x} = |\mathbf{x}|^{-1}\lambda(|\mathbf{x}|^{-1}\mathbf{x})\mathbf{x}.$$

The function r defined above is therefore continuous on the closure of $\mathbb{R}^n \setminus X$. It is obviously continuous on X itself. It follows that $r: \mathbb{R}^n \to X$ is continuous. This proves the result in the case when the topological interior of the set X is non-empty.

We now extend the result to the case where the topological interior of X is empty. Now the number of points in an affinely independent list of points of $\mathbb{R}^n$ cannot exceed $n+1$. It follows that there exists an integer q not exceeding n such that the convex set X contains a $q+1$ affinely independent points but does not contain $q+1$ affinely independent points. Let $\mathbf{w}_0, \mathbf{w}_1, \ldots, \mathbf{w}_q$ be affinely independent points of X. Let V be the q-dimensional subspace of $\mathbb{R}^n$ spanned by the vectors

$$\mathbf{w}_1 - \mathbf{w}_0, \mathbf{w}_2 - \mathbf{w}_0, \ldots, \mathbf{w}_q - \mathbf{w}_0.$$

Now if there were to exist a point $\mathbf{x}$ of X for which $\mathbf{x} - \mathbf{w}_0 \notin V$ then the points $\mathbf{w}_0, \mathbf{w}_1, \ldots, \mathbf{w}_q, \mathbf{x}$ would be affinely independent. The definition of q ensures that this is not the case. Thus if

$$X_V = \{\mathbf{x} - \mathbf{w}_0 : \mathbf{x} \in X\}.$$

then $X_V \subset V$. Moreover X_V is a closed convex subset of V.

Now it follows from Lemma 3.5 that the convex set X_V contains the q-simplex with vertices

$$\mathbf{0},\ \mathbf{w}_1 - \mathbf{w}_0,\ \mathbf{w}_2 - \mathbf{w}_0, \ldots\ \mathbf{w}_q - \mathbf{w}_0.$$

This q-simplex has non-empty topological interior with respect to the vector space V. It follows that X_V has non-empty topological interior with respect to V. It therefore follows from the result already proved that there exists a continuous function $r_V\colon V \to X_V$ that satisfies $r_V(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in X_V$. Basic linear algebra ensures the existence of a linear transformation $T\colon \mathbb{R}^n \to V$ satisfying $T(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in V$. Let

$$r(\mathbf{x}) = r_V(T(\mathbf{x} - \mathbf{w}_0)) + \mathbf{w}_0$$

for all $\mathbf{x} \in \mathbb{R}^n$. Then the function $r\colon \mathbb{R}^n \to X$ is continuous, and $r(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in X$, as required. ∎