MA3486—Fixed Point Theorems and Economic Equilibria
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3. Simplices and Convexity

3.1. Affine Independence

Definition

Points $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ in some Euclidean space $\mathbb{R}^k$ are said to be *affinely independent* (or *geometrically independent*) if the only solution of the linear system

$$\begin{cases} \sum_{j=0}^{q} s_j \mathbf{v}_j & = & \mathbf{0}, \\ \sum_{j=0}^{q} s_j & = & 0 \end{cases}$$

is the trivial solution $s_0 = s_1 = \cdots = s_q = 0$.

Lemma 3.1

Let $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ be points of Euclidean space $\mathbb{R}^k$ of dimension k. Then the points $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ are affinely independent if and only if the displacement vectors $\mathbf{v}_1 - \mathbf{v}_0, \mathbf{v}_2 - \mathbf{v}_0, \ldots, \mathbf{v}_q - \mathbf{v}_0$ are linearly independent.

Proof

Suppose that the points $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ are affinely independent. Let $s_1, s_2, \ldots, s_q$ be real numbers which satisfy the equation

$$\sum_{j=1}^{q} s_j(\mathbf{v}_j - \mathbf{v}_0) = \mathbf{0}.$$

Then $\sum_{j=0}^{q} s_j \mathbf{v}_j = \mathbf{0}$ and $\sum_{j=0}^{q} s_j = 0$, where $s_0 = -\sum_{j=1}^{q} s_j$, and therefore

$$s_0 = s_1 = \cdots = s_q = 0.$$

It follows that the displacement vectors $\mathbf{v}_1 - \mathbf{v}_0, \mathbf{v}_2 - \mathbf{v}_0, \ldots, \mathbf{v}_q - \mathbf{v}_0$ are linearly independent.

Conversely, suppose that these displacement vectors are linearly independent. Let $s_0, s_1, s_2, \ldots, s_q$ be real numbers which satisfy the equations $\sum_{j=0}^{q} s_j \mathbf{v}_j = \mathbf{0}$ and $\sum_{j=0}^{q} s_j = 0$. Then $s_0 = -\sum_{j=1}^{q} s_j$, and therefore

$$\mathbf{0} = \sum_{j=0}^{q} s_j \mathbf{v}_j = s_0 \mathbf{v}_0 + \sum_{j=1}^{q} s_j \mathbf{v}_j = \sum_{j=1}^{q} s_j (\mathbf{v}_j - \mathbf{v}_0).$$

It follows from the linear independence of the displacement vectors $\mathbf{v}_j - \mathbf{v}_0$ for $j = 1, 2, \ldots, q$ that

$$s_1 = s_2 = \cdots = s_q = 0.$$

But then $s_0 = 0$ also, because $s_0 = -\sum_{j=1}^{q} s_j$. It follows that the points $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ are affinely independent, as required. ∎

It follows from Lemma 3.1 that any set of affinely independent points in $\mathbb{R}^k$ has at most $k+1$ elements. Moreover if a set consists of affinely independent points in $\mathbb{R}^k$, then so does every subset of that set.

3.2. Simplices in Euclidean Spaces

Definition

A *q-simplex* in $\mathbb{R}^k$ is defined to be a set of the form

$$\left\{ \sum_{j=0}^{q} t_j \mathbf{v}_j : 0 \leq t_j \leq 1 \text{ for } j = 0, 1, \ldots, q \text{ and } \sum_{j=0}^{q} t_j = 1 \right\},$$

where $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ are affinely independent points of $\mathbb{R}^k$. These points are referred to as the *vertices* of the simplex. The non-negative integer q is referred to as the *dimension* of the simplex. (Thus a simplex of dimension q has $q + 1$ vertices.)

Example

A 0-simplex in a Euclidean space $\mathbb{R}^k$ is a single point of that space.

Example

A 1-simplex in a Euclidean space $\mathbb{R}^k$ of dimension at least one is a line segment in that space. Indeed let λ be a 1-simplex in $\mathbb{R}^k$ with vertices $\mathbf{v}$ and $\mathbf{w}$. Then

$$\begin{aligned}\lambda &= \{s\,\mathbf{v} + t\,\mathbf{w} : 0 \leq s \leq 1,\ 0 \leq t \leq 1 \text{ and } s + t = 1\} \\ &= \{(1-t)\mathbf{v} + t\,\mathbf{w} : 0 \leq t \leq 1\},\end{aligned}$$

and thus λ is a line segment in $\mathbb{R}^k$ with endpoints $\mathbf{v}$ and $\mathbf{w}$.

Example

A 2-simplex in a Euclidean space $\mathbb{R}^k$ of dimension at least two is a triangle in that space. Indeed let τ be a 2-simplex in $\mathbb{R}^k$ with vertices $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$. Then

$$\tau = \{r\,\mathbf{u} + s\,\mathbf{v} + t\,\mathbf{w} : 0 \leq r, s, t \leq 1 \text{ and } r + s + t = 1\}.$$

Let $\mathbf{x} \in \tau$. Then there exist $r, s, t \in [0, 1]$ such that $\mathbf{x} = r\,\mathbf{u} + s\,\mathbf{v} + t\,\mathbf{w}$ and $r + s + t = 1$. If $r = 1$ then $\mathbf{x} = \mathbf{u}$. Suppose that $r < 1$. Then

$$\mathbf{x} = r\,\mathbf{u} + (1 - r)\Big((1 - p)\mathbf{v} + p\mathbf{w}\Big)$$

where $p = \dfrac{t}{1 - r}$. Moreover $0 \leq r < 1$ and $0 \leq p \leq 1$. Also the above formula determines a point of the 2-simplex τ for each pair of real numbers r and p satisfying $0 \leq r \leq 1$ and $0 \leq p \leq 1$.

Thus

$$\tau = \left\{ r\,\mathbf{u} + (1-r)\Big((1-p)\mathbf{v} + p\mathbf{w}\Big) : 0 \leq p, r \leq 1. \right\}.$$

Now the point $(1-p)\mathbf{v} + p\mathbf{w}$ traverses the line segment $\mathbf{v}\,\mathbf{w}$ from $\mathbf{v}$ to $\mathbf{w}$ as p increases from 0 to 1. It follows that τ is the set of points that lie on line segments with one endpoint at $\mathbf{u}$ and the other at some point of the line segment $\mathbf{v}\,\mathbf{w}$. This set of points is thus a triangle with vertices $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$.

Example

A 3-simplex in a Euclidean space $\mathbb{R}^k$ of dimension at least three is a tetrahedron on that space. Indeed let $\mathbf{x}$ be a point of a 3-simplex σ in $\mathbb{R}^3$ with vertices $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$. Then there exist non-negative real numbers s, t, u and v such that

$$\mathbf{x} = s\,\mathbf{a} + t\,\mathbf{b} + u\,\mathbf{c} + v\,\mathbf{d},$$

and $s + t + u + v = 1$. These real numbers s, t, u and v all have values between 0 and 1, and moreover $0 \leq t \leq 1 - s$, $0 \leq u \leq 1 - s$ and $0 \leq v \leq 1 - s$. Suppose that $\mathbf{x} \neq \mathbf{a}$. Then $0 \leq s < 1$ and $\mathbf{x} = s\,\mathbf{a} + (1 - s)\mathbf{y}$, where

$$\mathbf{y} = \frac{t}{1-s}\,\mathbf{b} + \frac{u}{1-s}\,\mathbf{c} + \frac{v}{1-s}\,\mathbf{d}.$$

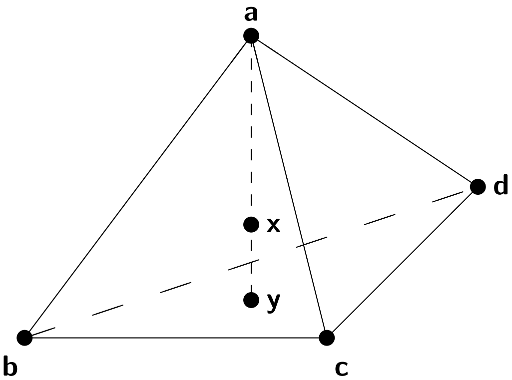
Moreover **y** is a point of the triangle **b c d**, because

$$0 \leq \frac{t}{1-s} \leq 1, \quad 0 \leq \frac{u}{1-s} \leq 1, \quad 0 \leq \frac{v}{1-s} \leq 1$$

and

$$\frac{t}{1-s} + \frac{u}{1-s} + \frac{v}{1-s} = 1.$$

It follows that the point **x** lies on a line segment with one endpoint at the vertex **a** of the 3-simplex and the other at some point **y** of the triangle **b c d**. Thus the 3-simplex σ has the form of a tetrahedron (i.e., it has the form of a pyramid on a triangular base **b c d** with apex **a**).


a
d
x
y
b
c

A simplex of dimension q in $\mathbb{R}^k$ determines a subset of $\mathbb{R}^k$ that is a translate of a q-dimensional vector subspace of $\mathbb{R}^k$. Indeed let the points $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ be the vertices of a q-dimensional simplex σ in $\mathbb{R}^k$. Then these points are affinely independent. It follows from Lemma 3.1 that the displacement vectors

$$\mathbf{v}_1 - \mathbf{v}_0, \mathbf{v}_2 - \mathbf{v}_0, \ldots, \mathbf{v}_q - \mathbf{v}_0$$

are linearly independent. These vectors therefore span a q-dimensional vector subspace V of $\mathbb{R}^k$. Now, given any point $\mathbf{x}$ of σ, there exist real numbers $t_0, t_1, \ldots, t_q$ such that $0 \leq t_j \leq 1$ for $j = 0, 1, \ldots, q$, $\sum_{j=0}^{q} t_j = 1$ and $\mathbf{x} = \sum_{j=0}^{q} t_j \mathbf{v}_j$. Then

$$\mathbf{x} = \left(\sum_{j=0}^{q} t_j \right) \mathbf{v}_0 + \sum_{j=1}^{q} t_j (\mathbf{v}_j - \mathbf{v}_0) = \mathbf{v}_0 + \sum_{j=1}^{q} t_j (\mathbf{v}_j - \mathbf{v}_0).$$

It follows that

$$\sigma = \left\{ \mathbf{v}_0 + \sum_{j=1}^{q} t_j(\mathbf{v}_j - \mathbf{v}_0) : 0 \leq t_j \leq 1 \text{ for } j = 1, 2, \ldots, q \right.$$
$$\left. \text{and } \sum_{j=1}^{q} t_j \leq 1 \right\},$$

and therefore $\sigma \subset \mathbf{v_0} + V$. Moreover the q-dimensional vector subspace V of $\mathbb{R}^k$ is the unique q-dimensional vector subspace of $\mathbb{R}^k$ that contains the displacement vectors between each pair of points belonging to the simplex σ.

3.3. Faces of Simplices

Definition

Let σ and τ be simplices in $\mathbb{R}^k$. We say that τ is a *face* of σ if the set of vertices of τ is a subset of the set of vertices of σ. A face of σ is said to be a *proper face* if it is not equal to σ itself. An r-dimensional face of σ is referred to as an *r-face* of σ. A 1-dimensional face of σ is referred to as an *edge* of σ.

Note that any simplex is a face of itself. Also the vertices and edges of any simplex are by definition faces of the simplex.

3.4. Barycentric Coordinates on a Simplex

Let σ be a q-simplex in $\mathbb{R}^k$ with vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$. If $\mathbf{x}$ is a point of σ then there exist real numbers $t_0, t_1, \ldots, t_q$ such that

$$\sum_{j=0}^{q} t_j \mathbf{v}_j = \mathbf{x}, \quad \sum_{j=0}^{q} t_j = 1 \text{ and } 0 \leq t_j \leq 1 \text{ for } j = 0, 1, \ldots, q.$$

Moreover $t_0, t_1, \ldots, t_q$ are uniquely determined: if $\sum_{j=0}^{q} s_j \mathbf{v}_j = \sum_{j=0}^{q} t_j \mathbf{v}_j$ and $\sum_{j=0}^{q} s_j = \sum_{j=0}^{q} t_j = 1$, then $\sum_{j=0}^{q} (t_j - s_j)\mathbf{v}_j = \mathbf{0}$ and $\sum_{j=0}^{q} (t_j - s_j) = 0$, and therefore $t_j - s_j = 0$ for $j = 0, 1, \ldots, q$, because the points $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ are affinely independent.

Definition

Let σ be a q-simplex in $\mathbb{R}^k$ with vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$, and let $\mathbf{x} \in \sigma$. The *barycentric coordinates* of the point $\mathbf{x}$ (with respect to the vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$) are the unique real numbers $t_0, t_1, \ldots, t_q$ for which

$$\sum_{j=0}^{q} t_j \mathbf{v}_j = \mathbf{x} \quad \text{and} \quad \sum_{j=0}^{q} t_j = 1.$$

The barycentric coordinates $t_0, t_1, \ldots, t_q$ of a point of a q-simplex satisfy the inequalities $0 \leq t_j \leq 1$ for $j = 0, 1, \ldots, q$.

Example

Consider the triangle τ in $\mathbb{R}^3$ with vertices at **i**, **j** and **k**, where

$$\mathbf{i} = (1, 0, 0), \quad \mathbf{j} = (0, 1, 0) \quad \text{and} \quad \mathbf{k} = (0, 0, 1).$$

Then

$$\tau = \{(x, y, z) \in \mathbb{R}^3 : 0 \leq x, y, z \leq 1 \text{ and } x + y + z = 1\}.$$

The barycentric coordinates on this triangle τ then coincide with the Cartesian coordinates x, y and z, because

$$(x, y, z) = x\,\mathbf{i} + y\,\mathbf{j} + z\mathbf{k}$$

for all $(x, y, z) \in \tau$.

Example

Consider the triangle in $\mathbb{R}^2$ with vertices at $(0,0)$, $(1,0)$ and $(0,1)$. This triangle is the set

$$\{(x,y) \in \mathbb{R}^2 : x \geq 0, \;\; y \geq 0 \text{ and } x + y \leq 1.\}.$$

The barycentric coordinates of a point (x,y) of this triangle are t_0, t_1 and t_2, where

$$t_0 = 1 - x - y, \quad t_1 = x \quad \text{and} \quad t_2 = y.$$

Example

Consider the triangle in $\mathbb{R}^2$ with vertices at $(1,2)$, $(3,3)$ and $(4,5)$. Let t_0, t_1 and t_2 be the barycentric coordinates of a point (x,y) of this triangle. Then t_0, t_1, t_2 are non-negative real numbers, and $t_0 + t_1 + t_2 = 1$. Moreover

$$(x,y) = (1 - t_1 - t_2)(1,2) + t_1(3,3) + t_2(4,5),$$

and thus

$$x = 1 + 2t_1 + 3t_2 \quad \text{and} \quad y = 2 + t_1 + 3t_2.$$

It follows that

$$t_1 = x - y + 1 \quad \text{and} \quad t_2 = \tfrac{1}{3}(x - 1 - 2t_1) = \tfrac{2}{3}y - \tfrac{1}{3}x - 1,$$

and therefore

$$t_0 = 1 - t_1 - t_2 = \tfrac{1}{3}y - \tfrac{2}{3}x + 1.$$

In order to verify these formulae it suffices to note that $(t_0, t_1, t_2) = (1, 0, 0)$ when $(x, y) = (1, 2)$, $(t_0, t_1, t_2) = (0, 1, 0)$ when $(x, y) = (3, 3)$ and $(t_0, t_1, t_2) = (0, 0, 1)$ when $(x, y) = (4, 5)$.